MS323 Block I Units 2–4

The Open University

Mathematics/Science:
An Interfaculty Third Level Course

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BLOCK I
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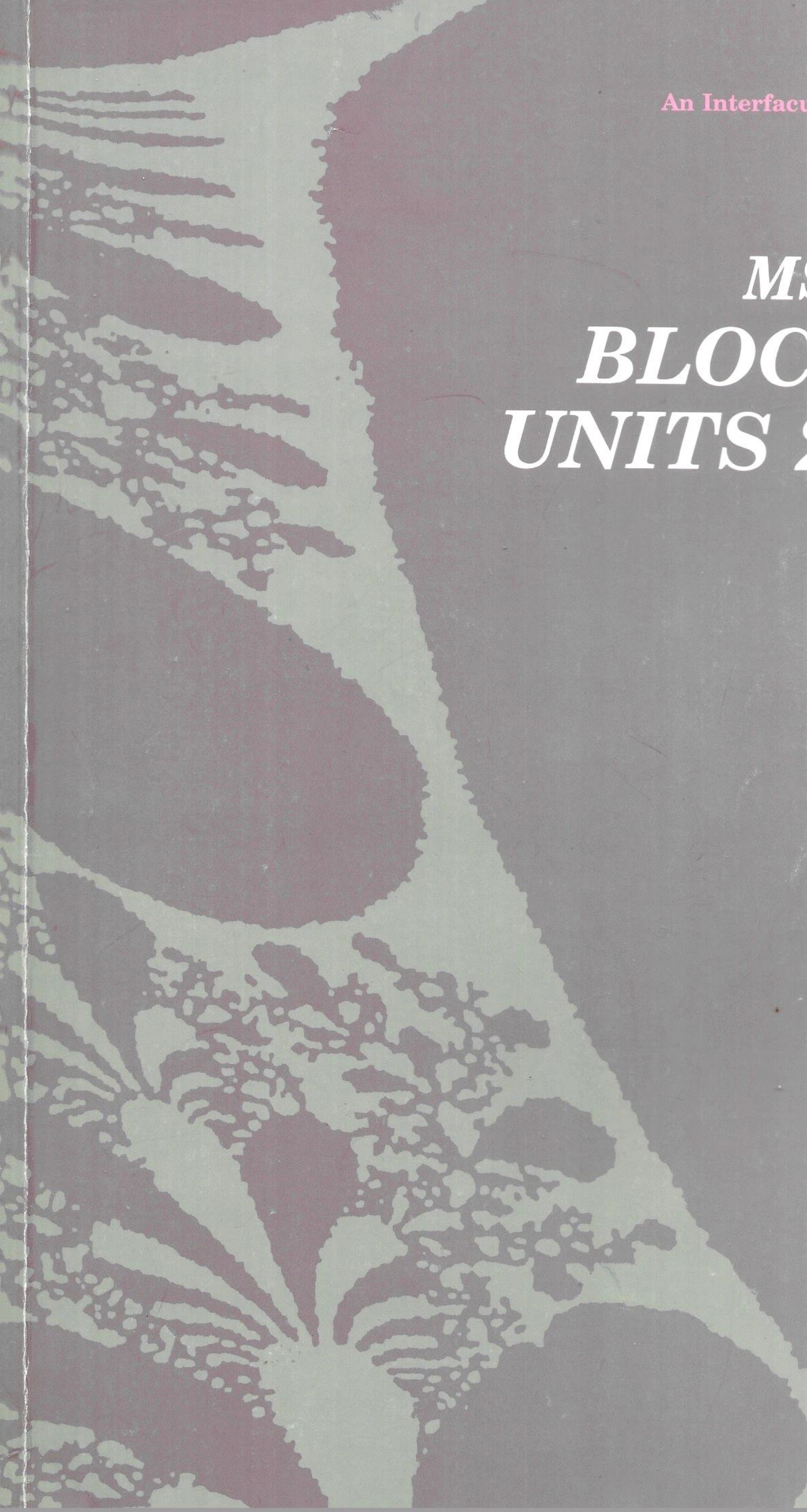
INTRODUCTION TO NON-LINEAR DYNAMICS



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INTRODUCTION TO NON-LINEAR DYNAMICS

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The Open University, Walton Hall, Milton Keynes, MK7 6AA.

First published 1991. New edition 1993.

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Edited, designed and typeset by the Open University using the Open University TEX System.

Printed in the United Kingdom by Page Bros, Norwich.

ISBN 0 7492 4777 0

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UNIT 2

FIRST-ORDER AUTONOMOUS SYSTEMS

2.1 The basic concepts of dynamics

We gave some preliminary attention to the subject matter of this course in the introduction to *Unit 1*; here we shall introduce the fundamental ideas in more detail.

Dynamical system

Dynamics is concerned with the motion of systems, that is, their change of state with time. The term *system*, sometimes extended to *dynamical system*, simply means the object or objects whose changes are under investigation. It is a portmanteau term which means different things in different contexts. For example, it may refer to a physical object like the Earth or the Moon, a collection of such objects as in the expression 'the solar system', or a representation of a physical system in terms of points in some space or in terms of differential equations. Generally speaking we shall be interested less in the real-world aspects of systems than in their mathematical properties, and so when we use the term 'system' we shall normally have some form of mathematical representation in mind. The term 'physical system' is a natural one to adopt for the real-world counterpart, though the use of the word 'physical' may seem to imply that the relevant science is physics, which is not necessarily the case.

State of a system

The essential first step in finding a mathematical representation of a physical system is to define one or more variables, or parameters, which correspond to the relevant features of the system: typical examples are the size of a population, the position of a particle, the mass of a chemical being produced in some reaction. A *state* of the system is determined by giving these variables specific values.

Often the range of possible values of such a variable is limited; for instance, a variable representing a mass can take only non-negative values. Any such limitations are usually readily taken into account.

There is clearly some ambiguity in the specification of states and state variables for any specific physical system, but in choosing the state variables it is essential that there are enough variables to determine past and future states, if the present state is known. To put it another way, the theory is deterministic.

State space

Phase space

The variables defining the states of a system may conveniently be thought of as coordinates in a space: if there is one variable then this space may be a line, if there are two variables it may be a plane, and so on. We shall also consider other spaces which may be given coordinates, such as the circle. The space of the state variables is named the *state space* or the *phase space* of the system; in this course we shall use the latter name. The term 'phase space' appears to have first been used by J. W. Gibbs (1839–1903) in his studies on the statistics of many-body systems:

when used in this context the term 'phase' has nothing to do with its use in the context of sinusoidal functions.

Each point in phase space represents a state of the system. The number of coordinates required to specify a point of phase space, which is the dimension of the space, is called the *order* of the system. It may appear that there is considerable possibility of confusion here, since the word 'order' already has several different meanings in mathematics. However, the main alternative use of the term in this course is in the context of differential equations, and, as we have already pointed out in *Unit 1*, we shall be treating second-order differential equations as systems of first-order equations in two variables, in other words as second-order systems, so the two uses will not be contradictory.

Order of a system

The time development of a system, whatever its nature, can therefore be considered as the motion of a representative point in phase space. We shall assume that the motion is governed by laws, which may, for example, be derived from a relevant scientific theory; how those laws are derived will not concern us very much. The form the laws take will, however, be of significance. As we mentioned in *Unit 1*, there are basically two types of system: continuous and discrete. We shall be concerned mainly with continuous systems in this course. The state of a system of order n is a point with coordinates $(x_1, x_2, \ldots, x_n)$ in an n-dimensional phase space. Motion of the continuous system gives rise to a curve in phase space, which will be defined by specifying the coordinates as functions of the time t, giving a time-dependent vector $(x_1(t), x_2(t), \ldots, x_n(t))$. We shall assume that the laws governing the motion lead to ***equations of motion*** or ***equations of change*** which form a system of n coupled first-order differential equations

Equations of motion

Equations of change

$$\begin{aligned}
\frac{dx_1}{dt} &= v_1(x_1, x_2, \ldots, x_n, t),\\
\frac{dx_2}{dt} &= v_2(x_1, x_2, \ldots, x_n, t),\\
&\vdots\\
\frac{dx_n}{dt} &= v_n(x_1, x_2, \ldots, x_n, t),
\end{aligned}$$

where the functions $v_1, v_2, \ldots, v_n$ may be functions of all the coordinates $x_1, x_2, \ldots, x_n$ and the time t. Given that second-order, and indeed higher-order differential equations may be replaced by equivalent systems of first-order ones, this assumption is not much of a restriction: in fact, all types of continuous dynamical system in which the motion is governed by differential equations – which is commonly the case – are covered by it.

The functions $v_1, v_2, \ldots, v_n$ are the components of a vector called the ***velocity field*** of the system; its value for particular $x_1, x_2, \ldots, x_n$ and t is the ***phase velocity***. Unless we say otherwise, it should be assumed that the functions $v_1, v_2, \ldots, v_n$ are well-behaved functions of their arguments, so that they may be differentiated and integrated at will. It is a property of such a system of differential equations that if values of the variables $x_1, x_2, \ldots, x_n$ and t are specified, then there is a unique solution of the system of equations for which $x_1, x_2, \ldots, x_n$ take their given values for the given value of t. This important property is the conclusion of a theorem known as the ***existence and uniqueness*** theorem for solutions of systems of first-order differential equations. One consequence of this property is that if we know the present state of the system (the values of the xs for a given value of t) then past and future states of the system may, in principle, be determined. The criterion that the system be deterministic is therefore satisfied.

Velocity field

Phase velocity

Existence and uniqueness of solutions of systems of first-order differential equations

Existence and uniqueness theorems are not part of this course, but for completeness we provide statements of the appropriate theorems in the Handbook.

An important special class of continuous dynamical systems consists of those for which the velocity field does not depend explicitly on time, so that the functions $v_1, v_2, \ldots, v_n$ are functions of the xs alone. Such systems are called ***autonomous*** (which just means time-independent) systems. Consider, for example, a damped

Autonomous system

harmonic oscillator, whose equations of motion may be written, in the manner explained in *Unit 1*, as a pair of first-order differential equations

$$\dot{x}_1 = v_1 = \frac{x_2}{m},$$
$$\dot{x}_2 = v_2 = -kx_1 - \frac{r}{m}x_2,$$

where m, k and r are constants representing the mass of the particle, the spring constant and the dashpot constant respectively. This is an example of an autonomous system. If, however, the oscillator is forced, by the imposition of an external sinusoidal force, then the equations of motion will contain a corresponding term which depends explicitly on time: for example, they may become

$$\dot{x}_1 = \frac{x_2}{m},$$
$$\dot{x}_2 = -kx_1 - \frac{r}{m}x_2 + A\cos\omega t,$$

where A and ω are constants. The system is no longer autonomous.

2.2 First-order systems: basic theory

First-order systems are amongst the simplest systems. A first-order system is characterised by the property that its states can be represented by a single real variable, x(there is no need to use a suffix when there is only one variable), which may be considered as a coordinate of a point in a one-dimensional phase space. Examples of one-dimensional systems are the population of a species, the ratio of Carbon-14 to Carbon-12 in an archaeological sample, the electrical charge in a condenser and the concentration of a constituent in a chemical reaction.

We shall assume to begin with that the phase space of the system is a line. In the case of a first-order system there are, in fact, only two essentially different possible phase spaces, namely the line and the circle. The circle is the appropriate phase space for a system for which the coordinate labelling the states is periodic, in the sense that there is some fixed non-zero number T such that for all x the points x and $x + T$ label the same state. Otherwise, the appropriate phase space is the line, which we can think of as an x-axis, and sometimes call the real line for emphasis. We postpone consideration of systems on the circle until towards the end of the unit, Subsection 2.4.3. In this section we shall describe the fundamental properties of first-order systems on the line.

The motion of a continuous first-order dynamical system can be represented by a function $x(t)$ of time satisfying a first-order differential equation

$$\frac{dx}{dt} = \dot{x} = v(x,t), \tag{2.1}$$

the equation of motion of the system. Here $v(x,t)$ is a sufficiently well-behaved *velocity function* of x and t, or velocity field, whose value for a particular x and t is the phase velocity. The motion of the system is prescribed once the state at any time is known, in accordance with the property of determinism described above.

Velocity function of a first-order system

We shall mostly restrict our attention to ***autonomous*** first-order systems, that is, systems which are not subject to any external influences that depend explicitly upon time. The equation of motion of an autonomous system then takes the form

Autonomous first-order system

$$\frac{dx}{dt} = \dot{x} = v(x). \tag{2.2}$$

The equation which governs the ratio of Carbon-14 to Carbon-12 in an archaeological sample, and which therefore lies at the basis of the carbon dating process, is $\dot{x} = -kx$, where k is a positive constant; this is therefore an example of an autonomous system. It is, of course, a vital factor in the use of the radioactive decay of carbon isotopes to date archaeological finds that the rate of decay k is not dependent on when the decay takes place; in other words, that the system should be autonomous.

Exercise 2.1

Three first-order systems have the following velocity functions. Which of them is autonomous?

(i) $v(x,t) = \exp x = e^x$

(ii) $v(x,t) = t$

(iii) $v(x,t) = \begin{cases} 0, & t < 0 \\ x^2, & t \geq 0 \end{cases}$

Exercise 2.2

In *Unit 1* we described the method of graphical representation of first-order differential equations based on direction fields. How can you tell from inspection of a direction field whether or not the differential equation it represents is autonomous?

Autonomous first-order systems are particularly simple because, for such systems, the velocity function is independent of time and the equation of motion is therefore immediately integrable, in principle at least. If x_0 represents the state at time t_0, then separation of the variables in Equation 2.2, followed by integration, gives t as a function of x:

$$\int_{x_0}^{x} \frac{dx}{v(x)} = \int_{t_0}^{t} dt = t - t_0. \tag{2.3}$$

Upon integration, Equation 2.3 gives the time as a function of x, $t(x)$. Normally we require the inverse function $x(t)$. Thus this method of solution requires that the integral can be evaluated in a form sufficiently convenient that the resulting equation can be solved for $x(t)$. In principle, both of these operations can be carried out provided that attention is restricted to ranges of values of x for which $v(x)$ is never zero, although in practice it may not be possible to give explicit expressions for either the integral or the inverse function.

In Equation 2.3 the solution of the equation of motion is represented in terms of definite integrals, with the variables x and t as the upper limits of integration and the specified particular values x_0 and t_0 as the lower limits. This is an alternative to the practice of solving a first-order differential equation by using indefinite integration, including a constant of integration, and evaluating that constant in terms of the specified values in order obtain a particular solution. It is clear that the solution obtained in Equation 2.3 satisfies $x = x_0$ when $t = t_0$, because each integral is zero when its upper limit equals its lower one. So this method has the advantage of combining the two steps, of integrating and evaluating the constant, into one. It must be used with care, however. In the first place, it might be assumed (in the case of the right-hand integral, for example) that since t is the upper limit, the solution holds only for $t \geq t_0$. This is not so; there is no reason why the upper limit of an integral should be numerically greater than the lower limit. Secondly, the integral on the left will make sense only so long as the range of x does not include any values for which $v(x) = 0$; thus it is implicitly assumed that $v(x_0) \neq 0$, and the solution holds in the largest interval around x_0 which contains no zeros of $v(x)$.

The pair of values, x_0 and t_0, used to specify the particular solution of the equation of motion, is often called an initial condition. This terminology is potentially misleading, for a reason we have just mentioned in relation to Equation 2.3: it suggests that we are interested only in what takes place after the instant of time $t = t_0$. In fact, we shall usually be concerned with the whole solution, that is to say, with the formula for x in terms of t for all values of t for which it makes sense. Even though the expression 'initial condition' has this drawback, we shall use it because it is convenient and standard; remember that it just means a pair of values of x and t that determine a particular solution of a system.

We shall often abbreviate 'solution of the equation of motion of a system' to 'solution of a system'.

Exercise 2.3

Solve the autonomous system

$$\dot{x} = bx,$$

where b is a constant. Give, as your answer, the solution which satisfies the initial condition $x = x_0$ when $t = t_0$. Be careful to distinguish between the cases $x_0 > 0$, $x_0 < 0$ and $x_0 = 0$; specify in each case the range of values of x for which the solution holds.

Exercise 2.4

Show that, for an autonomous system, the constant function $x(t) = c$ is a solution of the equation of motion if and only if c is a zero of the velocity function, that is, if and only if $v(c) = 0$.

Notice that in Exercise 2.3, the solution $x(t)$ depends upon t and t_0 only through the time interval $t - t_0$. It is clear from Equation 2.3 that the solution of the equation of motion of any autonomous system will have the same property.

Exercise 2.5

Show that the solution of the non-autonomous equation

$$\dot{x} = xt,$$

with initial condition $x = x_0$ when $t = t_0$, depends upon both t and t_0, not just $t - t_0$.

The fact that their solutions depend only on time differences is an important property of autonomous systems, and is restricted to them. It means that if the solution which satisfies the initial condition $x = x_0$ when $t = 0$ is known, then the solution which satisfies the initial condition $x = x_0$ when $t = t_0$ (same initial state, different time) may easily be found. Suppose that x is the solution satisfying the initial condition $x = x_0$ when $t = 0$. Then the solution $\tilde{x}$ which satisfies the initial condition $\tilde{x} = x_0$ at a different time $t = t_0$ is given by

$$\tilde{x}(t) = x(t - t_0). \tag{2.4}$$

To show this we shall appeal to the solution of Exercise 2.2, in which we showed that the direction field of the equation of motion for an autonomous system is invariant under translations parallel to the t-axis, or time-translations. Now if the direction field of a differential equation is unchanged by time-translation then clearly so are the solutions of the differential equation: that is to say, the effect of time-translating one solution of the equation is to give another solution of the same equation. The effect of a translation on a function is most easily seen by considering the effect of the translation on the function's graph.

A translation is a transformation in which each point is moved the same distance in the same direction.

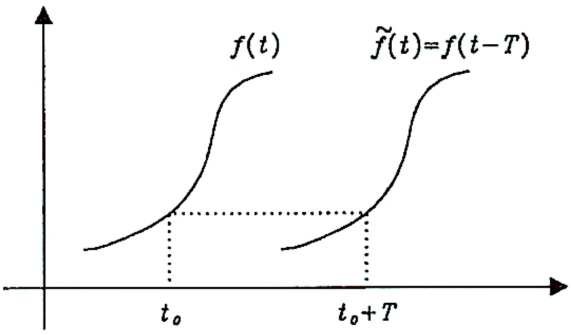


Figure 2.1 Translating a function's graph through a time interval T.

If $f(t)$ is a given function of t then the translate through T, say, of the graph of $f(t)$ is the graph of the function $\tilde{f}(t)$ such that

$$\tilde{f}(t + T) = f(t) \qquad \text{or} \qquad \tilde{f}(t) = f(t - T).$$

Thus if $x(t)$ is a solution of the equation of motion for the given autonomous system then so is $\tilde{x}(t) = x(t - t_0)$ as given in Equation 2.4; moreover,

$$\tilde{x}(t_0) = x(t_0 - t_0) = x(0) = x_0,$$

as required.

Uniqueness of solutions of first-order differential equations

This argument makes implicit use of the property of ***uniqueness of solutions*** of first-order differential equations, which is one of the results of the existence and uniqueness theorem mentioned in Section 2.1 applied in this simple case. The theorem states that there is, in general, only one solution of a first-order differential equation $\dot{x} = v(x, t)$ having a given value x_0 for a given value t_0 of t. This amounts, if you like, to saying that one initial condition is always sufficient for a first-order equation. We pointed out earlier the role this property has in relation to the criterion for a system to be deterministic. Here we use it for a different purpose.

Our argument above works on the principle that if it can be shown that a certain function satisfies the differential equation (which is done in this case by use of the invariance of solutions under time-translations), and if that function also satisfies some given initial condition, then it is the unique solution with that initial condition.

There is an alternative, more abstract, method of showing that the function $\tilde{x}(t)$ satisfies the differential equation, which involves simply differentiating it:

$$\frac{d\tilde{x}}{dt}(t) = \frac{dx}{dt}(t - t_0) = v(x(t - t_0)) = v(\tilde{x}(t)). \tag{2.5}$$

(Note that here $\frac{dx}{dt}(t - t_0)$ stands for the function obtained by first differentiating $x(t)$ with respect to t and then substituting $t - t_0$ for t in the result. In deriving Equation 2.5 we have used the chain rule and the fact that, since t_0 is constant, $d(t - t_0)/dt = 1$.) Thus $\tilde{x}(t)$ is a solution of the differential equation. In order for this to work it is clearly necessary that v does not depend explicitly on t.

Exercise 2.6

Let $x(t)$ be the solution of an autonomous first-order system which satisfies the initial condition $x = x_0$ when $t = t_0$. Show that the solution which satisfies the initial condition $x = x_0$ when $t = t_1$ is

$$x(t - (t_1 - t_0)).$$

Exercise 2.7

Let $x(t)$ be a solution of an autonomous first-order system and let s be a fixed number (which may be thought of as a value of the time). Show that the solution of the system which satisfies the initial condition $x = x(s)$ when $t = 0$ is

$$x(s + t).$$

Exercise 2.8

For a differential equation of the form $\dot{x} = v(t)$, where the right-hand side is a function of t alone (the very opposite of being autonomous!), the direction field is invariant under translations parallel to the x-axis. What familiar property of integration corresponds to the invariance of solutions under such translations?

The time-translation invariance of the solutions of autonomous systems is a consequence of any external influence being independent of time. For an autonomous system there is no natural choice of time origin, because the equations of motion and their solutions are unaffected by a time-translation, and therefore all time origins are equivalent. The imposition of a time-dependent external influence may determine a time origin, however – as the instant at which the influence is 'switched on', for example.

2.3 Fixed points, equilibrium and stability

In this section we shall show how certain simple properties of the graph of the velocity function $v(x)$ can be used to determine the qualitative features of the motion of the corresponding first-order autonomous system.

2.3.1 Graphical methods and fixed points

Equation 2.3 (page 8) gives a formal solution of the equation of motion, but even if the integral can be evaluated in terms of known functions, finding the required inverse function is rarely easy, and more often than not the integral can be evaluated only numerically. So a simple graphical technique for obtaining qualitative information about the motion, such as the one based on direction fields, is an important complement to other methods of dealing with the equations of motion. The direction field approach is unnecessarily complicated in this case, however, because the field of an autonomous system is time-translation invariant, and therefore contains the same information over and over again. In fact, it is enough to know the direction field along the x-axis to know it everywhere in the (x, t)-plane. We shall actually use a conceptually simpler, one-dimensional, method.

The idea is to draw at each point on the x-axis an arrow of length proportional to $|v(x)|$ (where x is the point's coordinate), based on the point and pointing in the positive direction if $v(x) > 0$, the negative direction if $v(x) < 0$. At points where $v(x) = 0$ one draws not an arrow, but simply a dot. The corresponding collection of arrows is a picture of the velocity field of the system. It is something like a direction field, but in one dimension, and the variations in the values of v are represented by the variations in the lengths of arrows rather than in the variations in the directions of line segments. Of course, we cannot physically draw an arrow at every point of the line, so to construct the picture in practice we choose some representative points. Since we are dealing with a first-order autonomous system we may easily draw the graph of $v(x)$ and use it to read off the information required to construct the arrows. It is often convenient to draw the graph and the arrows on the same diagram. We call the resulting diagram, with or without the graph of the velocity function, the *phase diagram* of the system.

Phase diagram

Example 2.1

We illustrate the method for the particular system $v(x) = -x + x^3$, with the equation of motion

$$\frac{dx}{dt} = -x(1 - x^2). \tag{2.6}$$

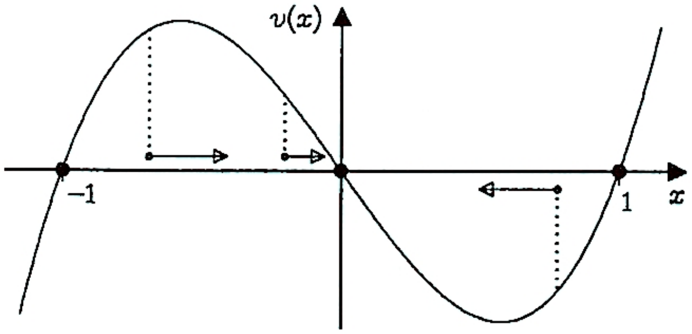


Figure 2.2 Graph of $v(x) = -x(1 - x^2)$, with arrows indicating the velocity field.

The graph of $v(x)$ is shown in Figure 2.2. Also shown is a set of arrows of varying lengths drawn parallel to the x-axis in the manner described above.

Since we think of $v(x)$ as being a velocity, the length of the arrow at x is proportional to the speed there – the phase speed. Using the velocity field we can describe in qualitative terms the motion of any phase point. For instance, a point initially at

$x = \frac{1}{2}$ is moving towards the origin, but moves slower and slower as it approaches the origin; we shall see below that the origin is not reached in any finite time. Similarly, a point initially at $x = -\frac{1}{2}$ is also moving towards the origin. But if initially $|x| > 1$, then the point is moving away from the origin, with phase speed increasing with $|x|$. Finally, if the initial point is at one of the zeros of $v(x)$, namely $x = 0, \pm 1$, then $\dot{x} = 0$ and the point stays where it is. Thus by drawing, or even just sketching, the graph of $v(x)$ we can obtain a qualitative picture of the whole motion. □

Example 2.7 on page 15

See Exercise 2.4, page 9.

It is clear that the zeros of the velocity field, that is, the roots of the equation

$$v(x) = 0, \tag{2.7}$$

are special. A system initially at a zero of $v(x)$ remains there for all time, so the zeros are named *fixed points* of the system. They are sometimes also called *equilibrium* points, and a system at a fixed point is said to be in equilibrium. At all other points the state of the system changes with time. Furthermore, a phase point initially on one side of a fixed point cannot pass to the other side of it during the motion of the system. For if this were to happen, the moving point would have to coincide with the fixed point at some time during the motion, and this is impossible because of the uniqueness of solutions of differential equations.

Fixed point of an autonomous first-order system

Equilibrium point

Usually the fixed points will be spaced out along the x-axis, and so divide the phase space into a collection of open intervals, the leftmost of which will extend from a fixed point to minus infinity, the rightmost from a fixed point to plus infinity, the others being finite in length. These intervals are invariant, in the sense that each is completely independent of all the others so far as the motion of the system is concerned. The same is true of the fixed points. Such fixed points and intervals are *invariant sets* of states, which are defined by the property that if the system is in such a set at some time it remains in that set for all time. More picturesquely, if we were to paint the invariant sets different colours initially, then there would be no mixing of colours during the subsequent motion. We usually consider only those *elementary* invariant sets which cannot be decomposed into smaller invariant sets.

Invariant set of states

Elementary invariant set

We shall use the following notation for open intervals. For any pair of numbers a, b with $a < b$, the *open interval* from a to b, in other words the set of points x on the line such that $a < x < b$, will be denoted by (a, b). We shall use $(-\infty, a)$ to denote the infinite interval to the left of a, consisting of the points x such that $x < a$; and similarly (a, ∞) for the infinite interval to the right of a, whose elements satisfy $x > a$.

Notation for open intervals

The phase space of a system may be regarded as a collection of invariant sets, and they provide valuable information about the system's behaviour over arbitrarily long periods of time.

Example 2.2

The system $v(x) = -x(1 - x^2)$ illustrated in Figure 2.2 has three fixed points, at $x = 0, \pm 1$, and four elementary one-dimensional invariant sets, which are the intervals $(-\infty, -1)$, $(-1, 0)$, $(0, 1)$, $(1, \infty)$, each bounded on one side or both sides by the fixed points. □

In the example above the fixed points are finite in number. There are other possibilities, of which the following two examples illustrate extreme cases.

Example 2.3

For the the zero velocity field $v(x) = 0$, every point is a fixed point; the system is always and everywhere at rest,

$$x(t) = x_0,$$

and so every phase point and every set of phase points is an invariant set. □

Example 2.4

For the constant, or uniform, velocity field $v(x) = a$, $a \neq 0$, there are no fixed points and the only invariant set is the entire phase space. A sample phase diagram is shown below.

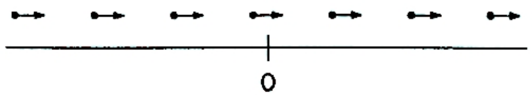


Figure 2.3 Phase diagram for a uniform velocity field, $v(x) = a$, with $a > 0$.

The solution of this system with initial condition $x = x_0$ at $t = 0$ is

$$x(t) = x_0 + at. \qquad \square$$

Exercise 2.9

Sketch the phase diagrams and find the fixed points and invariant sets of the systems with the following velocity functions. Here a and b are any pair of numbers, with $b > a$.

(i) $v(x) = (x - a)(x - b)$

(ii) $v(x) = (a - x)(x - b)$

Exercise 2.10

Sketch the phase diagram and find the fixed points and invariant sets of the system with the velocity function $v(x) = \sin x$.

The analysis of the motion of a dynamical system as the solution of differential equations emphasises the motion of each individual phase point as a function of time. However, it is often advantageous to take a different view, and consider the effect on the phase space as a whole of letting the system run for a particular time. To make this clearer, let us choose an initial time, say $t = 0$ for definiteness – for autonomous systems the choice of initial time is irrelevant because of the time-translation invariance of solutions. Denote by $\phi(x, t)$ the state at time t of the system if it is initially in the state x. The function $\phi(x, t)$ is, of course, a function of two variables. By its definition,

$$\phi(x, 0) = x. \tag{2.8}$$

The motion of a particular phase point is the function of t obtained by keeping x fixed in the function $\phi(x, t)$; we shall write this as $\phi_x(t)$, the subscript x indicating that we are dealing with the 'partial function' obtained by fixing x. This corresponds to the point of view in which one concentrates on the motion of an individual point as a function of time. The alternative is to look at the transformation of the phase space as a whole obtained by letting the system evolve for a fixed time t; this corresponds to taking the other partial function, which is a function of x and is denoted by $\phi_t(x)$.

There is a helpful analogy for this new attitude, in which the dynamical system is likened to the steady flow of a liquid in a channel (the phase space). The velocity $v(x)$ represents the direction and speed of flow of the particle of liquid currently at the point x. Suppose that individual particles of the liquid could be labelled in some way, so as to identify them – by painting each a different colour, for example; and that we were to take two colour snapshots of the liquid, one at time 0, the other at a later time t. Then the partial function $\phi_t(x)$ would record the transformation from the first snapshot to the second.

We shall also allow t to be negative, so that $\phi_t(x)$ represents a state at a time earlier than 0, namely the state which evolves into the state x as time runs on from t to 0.

The analogy between the dynamical system and the flowing liquid helps to explain the terminology: for each t, the function ϕ_t, which is considered as a transformation of the phase space, is called the *flow* or *phase flow* of the dynamical system through time t. It is important to remember that we are dealing with autonomous first-order systems here; the concept of a flow cannot be defined directly for non-autonomous systems, and is a little more complicated for higher-order systems.

Flow (or phase flow) of an autonomous first-order dynamical system

Since we consider the flow as a transformation of phase space, we may examine its effect on subsets of the space. If S is any subset of phase space (for example, an open interval) then $\phi_t(S)$ is just the collection of the images of all the points of S under the flow, that is, the set of states $\phi_t(x)$ for x in S. Thus to say that S is invariant is to say that $\phi_t(S)$ is contained in S for all t.

Example 2.5

In the case of the uniform velocity field, $v(x) = a$, we have

This system was discussed in Example 2.4.

$$\phi_t(x) = x + at;$$

that is, the flow through time t is translation of the phase space through at. Note that, for $t \neq 0$, every point of the phase space moves under the flow; this is another way of saying that there are no invariant sets other than the whole space, as we remarked before. □

Example 2.6

For the velocity field $v(x) = bx$ the flow is given by

See Exercise 2.3, page 9.

$$\phi_t(x) = e^{bt}x.$$

As a transformation of the phase space this is a scaling by the factor e^{bt}. It leaves the origin fixed, maps the positive half of the line to itself, and likewise the negative half to itself. When $b > 0$, points other than the origin move away from the origin for positive t, since $e^{bt} > 1$, and towards it for negative t, when $0 < e^{bt} < 1$; for $b < 0$ these conclusions are reversed. □

Exercise 2.11

Define a fixed point of a dynamical system in terms of its flow.

Exercise 2.12

Show that the time-translation invariance property of autonomous first-order systems is equivalent to the following property of flows: for any two times s and t,

Exercise 2.7 (page 10) is relevant here.

$$\phi_s(\phi_t(x)) = \phi_{s+t}(x).$$

Confirm explicitly that this is the case for the systems discussed in Examples 2.5 and 2.6.

The result of Exercise 2.12 is often written $\phi_s \circ \phi_t = \phi_{s+t}$, which means that the effect of carrying out the transformation ϕ_t and following it with the transformation ϕ_s is the same as the effect of carrying out the single transformation ϕ_{s+t}. This is just like a time-translation – translation by t followed by translation by s is collectively the same as translation by $s + t$ – and so the flow may be interpreted as the carrying over to phase space of time-translations. (Also, for pure mathematics enthusiasts, the result shows that the transformations ϕ_t form a group under composition: it follows from Equation 2.8 that the identity element is ϕ_0, each element has an inverse (see Exercise 2.13 below), and the group multiplication is commutative. In fact, this group is homeomorphic to the group of real numbers under addition; for this reason a flow is sometimes called a one-parameter group.)

In the case of a constant velocity field the flow really is a translation, but not otherwise.

Exercise 2.13

Show that every transformation ϕ_t has an inverse, namely ϕ_{-t}.

2.3.2 Classification of fixed points

Simple fixed point

We shall now study the motion of a dynamical system near one of its fixed points; this involves the classification of fixed points into different types. It turns out that the analysis is most straightforward if the derivative of the velocity function does not vanish at the fixed point in question; when this is the case the fixed point is said to be *simple*. Thus a simple fixed point of an autonomous first-order dynamical system with velocity function $v(x)$ is a point x_f such that

$$v(x_f) = 0 \qquad \text{but} \qquad \frac{dv}{dx}(x_f) = v'(x_f) \neq 0.$$

Example 2.7

Motion near a linear fixed point

The simplest case of a simple fixed point occurs when $v(x)$ is a linear function of x, say $v(x) = a + bx$, where $b \neq 0$; there is a simple fixed point of this system at $x_f = -a/b$. We can rewrite the equation of motion as

$$\dot{x} = b(x - x_f). \tag{2.9}$$

This is essentially the same as the equation treated in Exercise 2.3 (page 9) and Example 2.6, and when $a = 0$ it is exactly the same. The motion in the latter case has already been described. The solution of the equation of motion,

$$\dot{x} = bx,$$

with initial condition $x = x_0$ when $t = 0$, is

$$x(t) = x_0 e^{bt}.$$

As we know from earlier discussions, the behaviour of the solution depends on the sign of b. If $b < 0$ then for any $x_0 \neq 0$, $x(t)$ tends to the fixed point, 0, as t tends to plus infinity; in this case all states (other than 0) approach 0 as time increases. If $b > 0$, on the other hand, then $x(t) \to +\infty$ as $t \to \infty$ if $x_0 > 0$, while $x(t) \to -\infty$ as $t \to \infty$ if $x_0 < 0$; all states (other than 0) leave 0 as time increases. In the first case, if the system is slightly disturbed from equilibrium it will tend to revert to it, and so the fixed point is said to be stable. In the second case, on the other hand, when equilibrium is once disturbed the system will not revert to its initial state, and the fixed point is unstable.

Notice that in the stable case, $b < 0$, if $x_0 \neq 0$ then $x(t) \neq 0$ for any finite time; so although the system tends towards the fixed point, it never actually reaches it.

The same general conclusions apply to the case $a \neq 0$, but in relation to the fixed point x_f, which is no longer at the origin. In fact, the results may be deduced directly by noting that the change of variable $\xi = x - x_f$ moves the fixed point to $\xi = 0$ and changes the velocity field to $v(\xi) = b\xi$. Then since $\dot{\xi} = \dot{x}$, the equation of motion is $\dot{\xi} = b\xi$, and the previous analysis holds with x replaced by ξ. Thus for $b < 0$, $x(t) \to x_f$ as $t \to \infty$; the fixed point is called stable, as before. For $b > 0$, on the other hand, all states other than x_f leave x_f as $t \to \infty$, and the fixed point is unstable.

The phase diagrams for the two cases are shown below. ☐

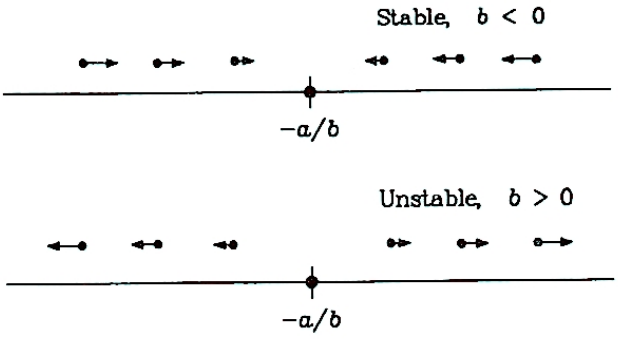


Figure 2.4 Phase diagrams for $\dot{x} = a + bx$.

Exercise 2.14

Show that the solution of the linear system $\dot{x} = a + bx$ with $b \neq 0$ is

$$x(t) = x_f + (x_0 - x_f)e^{bt},$$

where $x_0 = x(0)$ and x_f is the fixed point, and confirm the qualitative description of the motion given above.

This is a very important example as it describes the motion in a sufficiently small neighbourhood of any simple fixed point. To show that this is so, we make a first-order Taylor expansion of the velocity function $v(x)$ about a simple fixed point x_f to obtain

$$v(x) = (x - x_f)\,v'(x_f) + O\big((x - x_f)^2\big)\,, \tag{2.10}$$

where $v'(x_f)$, the derivative of $v(x)$ evaluated at x_f, is non-zero, by assumption. The motion near a simple fixed point becomes particularly easy to understand as the velocity function can be approximated by its linearised form

$$v(x) = (x - x_f)\,v'(x_f)$$

to obtain the easily soluble equation of motion discussed in Example 2.7 and Exercise 2.14 above. The motion of points sufficiently close to the fixed point is then well represented by the equation

$$\dot{x} = (x - x_f)\,v'(x_f) \tag{2.11}$$

with solution

$$x(t) = x_f + (x_0 - x_f)e^{bt}, \qquad \text{where} \quad b = v'(x_f). \tag{2.12}$$

Thus nearby points approach x_f, or leave it, as $t \to \infty$, according as $v'(x_f) < 0$ or $v'(x_f) > 0$.

We call Equation 2.11 the *linearised equation of motion* or the *linear approximation* to the equation of motion in the vicinity of the fixed point x_f.

Linearised equation of motion, linear approximation

The simple fixed points of any autonomous first-order dynamical system are thus of two types, corresponding to those found for linear systems. A fixed point x_f is *stable* if all sufficiently close neighbouring states (other than x_f itself) approach x_f as $t \to \infty$. A fixed point x_f is *unstable* if neighbouring states (other than x_f itself) leave x_f as $t \to \infty$.

Stable fixed point

Unstable fixed point

We have a straightforward test of whether a simple fixed point is stable or unstable, which requires nothing more than a knowledge of the sign of the derivative of the velocity function at the fixed point: if this is negative the fixed point is stable, if positive the fixed point is unstable. More generally, if $v(x)$ is a decreasing function of x near x_f then x_f is stable, while if $v(x)$ is increasing near x_f then x_f is unstable.

It is also easy to classify a fixed point from a phase diagram: if the arrows near the fixed point are directed towards it, as in the upper diagram in Figure 2.4, then the fixed point is stable; if they are directed away, as in the lower diagram in the same figure, then the point is unstable.

The two possibilities, and the various ways of distinguishing between them, are clearly illustrated by the graph of $v(x)$ near a simple fixed point, as shown in Figure 2.5.

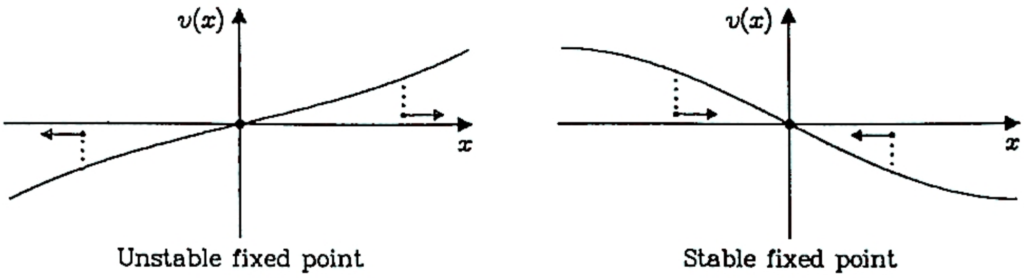


Figure 2.5 Typical velocity fields, $v(x)$, in the neighbourhoods of unstable and stable simple fixed points.

The use of the first-order Taylor expansion amounts to saying that locally the velocity function can be represented by a straight line through x_f with (non-zero) gradient $v'(x_f)$.

While it is enough to know the sign of $v'(x_f)$ to determine the nature of the fixed point x_f, the magnitude of $v'(x_f)$ can also be of interest. In the linear approximation to the motion near a simple fixed point x_f, phase points near x_f approach it or leave it exponentially. The magnitude of $v'(x_f)$ determines how quickly they do so: the larger $|v'(x_f)|$, the faster the exponential rate of approach or departure.

In the case of an unstable fixed point the linear approximation will hold only for small positive times, since every nearby phase point will soon move outside any interval in which the first-order Taylor expansion gives a good approximation to $v(x)$. The linear approximation will hold only for small negative times in the case of a stable fixed point, for similar reasons.

It is important to note that the definitions of stable and unstable fixed points depend crucially on the direction of time, in the sense that they are concerned with the behaviour of points near the fixed point as time increases. In fact, if one takes a system and transforms it to a new one by reversing the direction of time, then the stable fixed points of the old system become unstable fixed points of the new system, and the unstable ones of the old system become stable in the new.

Exercise 2.15

Show that all the fixed points of the dynamical system with velocity function $v(x) = -x(1 - x^2)$ are simple, and classify them as stable or unstable.

This system was discussed in Example 2.1, page 11.

Exercise 2.16

Find and classify the fixed points of the systems with the following velocity functions, where $b > a$.

(i) $v(x) = (x - a)(x - b)$
(ii) $v(x) = (a - x)(x - b)$

These systems were considered in Exercise 2.9, page 13.

Exercise 2.17

Consider an autonomous system which has exactly two simple fixed points. Can both fixed points be stable? Can both fixed points be unstable? How many essentially different phase diagrams can there be for such a system?

Exercise 2.18

Suppose that the system with velocity function $v(x)$ has a finite number of simple fixed points. Write down a velocity function for a system which has the same number of fixed points, but of opposite types to those of the given system, so that the new system has a stable fixed point where the given system has an unstable one, and the new system has an unstable fixed point where the given system has a stable one.

Not all fixed points are simple, and the classification of non-simple fixed points is rather more complicated than that for simple fixed points. For instance, if the velocity field is

$$v(x) = cx^2, \qquad c > 0, \tag{2.13}$$

then there is clearly a fixed point at $x = 0$, but $v'(0) = 0$ so it is not simple. As $v(x) > 0$ on both sides of $x = 0$, the phase diagram is as shown below.

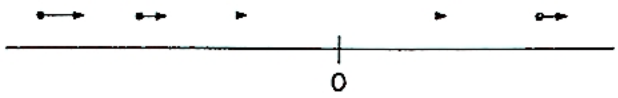


Figure 2.6 Phase diagram for $\dot{x} = cx^2$, $c > 0$.

The fixed point is neither stable nor unstable: neighbouring states approach it from the left, but leave it to the right.

Exercise 2.19

Draw a phase diagram, and describe qualitatively the motion of points near the fixed point 0, for the system with velocity field $v(x) = cx^2$, where $c < 0$.

Exercise 2.20

Suppose that x_f is a fixed point of a system with velocity function $v(x)$, that $v'(x_f) = 0$ but that $v''(x_f) \neq 0$. Discuss the nature of the fixed point x_f. [Hint: Use a Taylor expansion and the method adopted for simple fixed points in the paragraph containing Equation 2.10 (page 16).]

Exercise 2.21

Show that, unlike the case for a simple fixed point, for the fixed point discussed in Exercise 2.19 the motion of a nearby point is not exponential.

Exercise 2.22

By considering the velocity function $v(x) = dx^3$, where d is a non-zero constant, show that a fixed point which is not simple may nevertheless be stable or unstable. Show that, again, the motion of a point near the fixed point is not exponential in this case; but that nevertheless, when the fixed point is stable no other point can reach it in a finite time.

Exercise 2.23

Reconsider Exercise 2.17 without the assumption that the fixed points are simple. In particular, how many essentially different phase diagrams can there be for a system with two fixed points, not necessarily simple?

2.3.3 The effects of perturbations

There is another important difference between simple and non-simple fixed points, which concerns what happens if the velocity function is slightly changed.

Suppose that we take a linear velocity field, $v(x) = bx$, and vary it slightly by adding to it a 'perturbing' field $\varepsilon w(x)$, where $w(x)$ is a well-behaved function defined in a neighbourhood of $x = 0$, and ε is a constant, by implication small. It can be shown that, whatever the function $w(x)$, for sufficiently small $|\varepsilon|$ the new system, $\overline{v}(x) = bx + \varepsilon w(x)$, will continue to have a fixed point near to 0, which will be of the same nature as that of the original, linear, system. If the same thing is done to the field cx^2, on the other hand, the number and nature of the fixed points may change. The following exercises illustrate what may occur.

Exercise 2.24

The simplest case occurs when the perturbing function is constant; it must, of course, be non-zero if anything of interest is to occur. We may as well then take $w(x) = 1$.

Show that for sufficiently small $|\varepsilon|$ the nature of the fixed point of

$$\overline{v}(x) = bx + \varepsilon, \qquad b \neq 0,$$

is independent of ε. Show that, on the other hand, the velocity field

$$\overline{v}(x) = cx^2 + \varepsilon, \qquad c \neq 0,$$

has 2, 1 or 0 fixed points according as $\varepsilon/c < 0$, $\varepsilon = 0$ or $\varepsilon/c > 0$, and classify them.

Exercise 2.25

The system with the quadratic velocity function $\overline{v}(x) = cx^2 + \varepsilon x$, $c > 0$, has a fixed point at the origin. Describe how the nature of this fixed point depends on the value of ε.

The number and type of the fixed points of a dynamical system are important qualitative features of the flow of the system. In fact, for the usual kind of autonomous first-order system they determine the structure of the flow. Provided that the fixed points are isolated from each other, they divide the phase space into elementary invariant intervals, each consisting of points which move in the same direction under the flow. The *structure* of the flow can then be described by giving a list of the fixed points and of the directions of the flows in the corresponding elementary invariant intervals.

Structure of a flow

Some systems retain their structure under any sufficiently small perturbation: any system whose fixed points are all simple is an example. For other systems there are arbitrarily small perturbations that change the structure of the flow: the system with velocity function $v(x) = cx^2$ is an example, as the results of Exercises 2.24 and 2.25 show.

We normally assume that deterministic systems are robust, that is, not significantly affected by small perturbations, so that we generally expect models of these systems to be insignificantly changed by arbitrarily small changes.

2.4 Some special types of motion

Up to now we have considered only systems with the whole real line as phase space, and for the most part systems whose motions are defined for all values of the time. In this section we consider some cases of systems for which one or other of these conditions does not hold.

2.4.1 Terminating motion

The quadratic velocity function illustrates another important feature, terminating motion.

Exercise 2.26

For the velocity field $v(x) = cx^2$, with $c > 0$, show that the solution with initial condition $x = x_0 > 0$ at $t = 0$ is

$$x(t) = \frac{x_0}{1 - cx_0t}, \qquad t < (cx_0)^{-1}.$$

The motion terminates abruptly at the critical time $t = (cx_0)^{-1}$, when x tends to infinity, and is undefined beyond that time. Thus for some velocity functions the motion of the system may not be defined for all time. Of course, in physical applications the validity of the equation of motion itself usually breaks down before the critical time is reached.

In general, we refer to the motion as *terminating* if it is represented by a solution of the equation of motion which is undefined at some finite time. Note that the example of the quadratic velocity function shows that this phenomenon can happen for perfectly well-behaved, and indeed quite simple-looking, velocity functions. In such a case terminating motion occurs because x becomes infinite in a finite time. Motion can terminate in the past as well as the future. A situation which always leads to terminating motion arises whenever $v(x)$ is a polynomial of degree two or greater, with real roots (which are the fixed points of the system); motion in either of the two infinite invariant intervals is then bound to be terminating in either the future or the past.

Terminating motion

Exercise 2.27

Show that for all positive integers $n \geq 2$ the motion with velocity field

$$v(x) = x^n,$$

and with initial condition $x = x_0 \neq 0$ when $t = 0$, is terminating.

2.4.2 Natural boundaries

If the variable x represents a number of objects such as living cells or radioactive atoms, for example, then negative values of x have no meaning. We may then restrict the phase space to be the non-negative real line $x \geq 0$. The phase space is said to have a natural boundary at $x = 0$. Other physical limitations may put other constraints on the possible range of values of the variable x. In order to correspond to reality, the phase space may then have to be restricted to be a finite or infinite interval $[a, b]$. (Here, for a and b finite, $[a, b]$ denotes the *closed interval* $a \leq x \leq b$. It will be convenient to write $[-\infty, a]$ for the infinite interval $x \leq a$, which includes the end-point a, and $[a, \infty]$ for the infinite interval $x \geq a$.) Each of the points a and b, if finite, is called a *natural boundary* of the system.

Notation for closed intervals

Natural boundary

Notice that the boundary points are usually included in the phase space. The behaviour of the system near its boundary points therefore becomes a matter of interest. Normally, for first-order systems, the velocity at a natural boundary is zero, but this is not always the case.

Examples of first-order autonomous systems with natural boundaries may easily be constructed by taking any autonomous velocity field, defined and well-behaved on the whole line, which has a fixed point, and restricting the phase space to an invariant set bounded by and including the fixed point. For example, the system with the velocity field $v(x) = bx$, $b < 0$, and with phase space $[0, \infty]$ obtained by deleting the negative real axis $x < 0$, represents the exponential decay of a population, with a natural boundary at $x = 0$. Alternatively, one could take a velocity field which has two or more fixed points and restrict the phase space to be the closed interval between a pair of them. In either of these cases the behaviour of the system near a boundary point is easy to determine, in principle: it will be the same as the behaviour of the unrestricted 'parent' system on the appropriate side of the corresponding fixed point. Thus if the boundary point is a stable fixed point of the unrestricted system, for example, then nearby points of the restricted system will approach it as time increases; but they will never actually reach this boundary in a finite time.

There are, however, interesting examples of systems which have natural boundaries which cannot be obtained by restriction of a system defined and well-behaved on the whole line. One such example is the system with the equation of motion

$$\dot{x} = v(x) = -\sqrt{x} \tag{2.14}$$

(where $\sqrt{\ }$ means the positive square root). This system has a natural boundary at $x = 0$ and its phase space is the infinite interval $[0, \infty]$. But the velocity function $v(x) = -\sqrt{x}$ is not defined on any interval larger than $[0, \infty]$, and so this system cannot be obtained by restriction of a system defined and well-behaved on any larger interval. Furthermore, although the velocity function has a well-defined value at $x = 0$, it does not have a (finite) derivative there. Since the existence and uniqueness theorem for the solutions of first-order differential equations, applied to an equation of the form $\dot{x} = v(x)$, holds only for differentiable functions $v(x)$, we should not be surprised if solutions of the equation $\dot{x} = -\sqrt{x}$ behave peculiarly at $x = 0$.

The behaviour of the system whose equation of motion is Equation 2.14 is therefore not immediately obvious. It is clear that $v(0) = 0$, and that for $x_0 > 0$ the solution with initial condition x_0 always decreases towards 0 as t increases. However, this does not imply that the system behaves like one with a stable fixed point at 0, such as $\dot{x} = bx$.

The solution of Equation 2.14 for $x_0 > 0$ is given by

$$-\int_{x_0}^{x} \frac{dx}{\sqrt{x}} = \int_0^t dt \qquad \text{or} \qquad 2\left(\sqrt{x_0} - \sqrt{x}\right) = t,$$

whence

$$x(t) = \left(\sqrt{x_0} - \tfrac{1}{2}t\right)^2. \tag{2.15}$$

The system therefore reaches the natural boundary $x = 0$ at $t = 2\sqrt{x_0}$. Later than this, however, $x(t)$ defined by Equation 2.15 does not satisfy Equation 2.14, because

$$\dot{x} = -\left(\sqrt{x_0} - \tfrac{1}{2}t\right)$$

while

$$\sqrt{x} = \tfrac{1}{2}t - \sqrt{x_0} \qquad \text{for} \quad t > 2\sqrt{x_0}.$$

Thus for $t > 2\sqrt{x_0}$, $x(t)$ as given in Equation 2.15 is a solution of $\dot{x} = \sqrt{x}$, which has the opposite sign to the original equation of motion. No motion satisfying Equation 2.14 is possible beyond $t = 2\sqrt{x_0}$. Thus the boundary is reached in a finite time, and motion terminates there. This is quite different behaviour from that shown by the system describing the exponential decay of a population.

When examining the behaviour of a system near a natural boundary it is often helpful to approximate the velocity function near the boundary point. This is an extension of the technique we used in our discussion of stability of simple fixed points in Subsection 2.3.2. The following example illustrates the method.

Example 2.8

The system with velocity function

$$v(x) = \sqrt{1 - x^2}$$

is defined on the finite closed interval $[-1, 1]$. We shall obtain an approximate expression for the motion in the vicinity of the boundary point $x = 1$.

The graph of the velocity function $v(x) = \sqrt{1 - x^2}$ on $[-1, 1]$ is shown in Figure 2.7. In this interval $v(x) > 0$, so $x(t)$ is increasing. At $x = \pm 1$, $v(x) = 0$, but all derivatives of $v(x)$ are undefined, so as in the case already discussed, the solution does not behave like that near a simple fixed point.

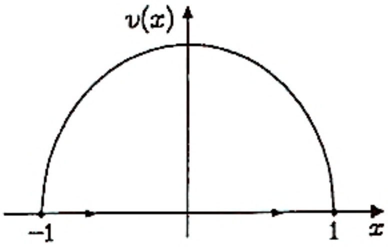


Figure 2.7 Graph of the velocity function $v(x) = \sqrt{1-x^2}$.

Let us consider the solutions near $x = 1$. We can express the velocity function as

$$v(x) = \sqrt{1-x^2} = \sqrt{1-x}\sqrt{1+x}.$$

The second factor on the right is well-behaved near $x = 1$; we may therefore Taylor expand it about $x = 1$, to obtain

$$v(x) = \sqrt{1-x}\left(\sqrt{2} + \frac{1}{2\sqrt{2}}(x-1) + O((x-1)^2)\right),$$

whence

$$v(x) = \sqrt{2}\sqrt{1-x} + O\left((x-1)^{3/2}\right).$$

Thus for x close to 1 (note that in the phase space $1 - x \geq 0$) we may approximate the equation of motion by

$$\dot{x} = \sqrt{2}\sqrt{1-x}. \tag{2.16}$$

This is similar to Equation 2.14. The similarity is more apparent if we make the substitution $\xi = 1 - x$, or $x = 1 - \xi$, so that $\dot{x} = -\dot{\xi}$; for then Equation 2.16 becomes

$$\dot{\xi} = -\sqrt{2}\sqrt{\xi}.$$

The solution is

$$\xi(t) = \left(\sqrt{\xi_0} - t/\sqrt{2}\right)^2,$$

or

$$x(t) = 1 - \left(\sqrt{1-x_0} - t/\sqrt{2}\right)^2.$$

Thus for x_0 sufficiently near the right-hand end of the interval $[-1, 1]$ a point initially at x_0 should reach the right-hand boundary point in a finite time, given approximately by $\sqrt{2(1-x_0)}$.

In cases like this the approximate equation of motion may be found by making an appropriate substitution right at the start, and this is sometimes easier. If this method had been adopted above we should have set $\xi = 1 - x$, giving

$$\begin{aligned}\sqrt{1-x^2} = \sqrt{1-(1-\xi)^2} &= \sqrt{2\xi - \xi^2} \\ &= \sqrt{2\xi}\sqrt{1 - \tfrac{1}{2}\xi} \\ &= \sqrt{2\xi}\left(1 - \tfrac{1}{4}\xi + O(\xi^2)\right) \\ &= \sqrt{2\xi} + O(\xi^{3/2}),\end{aligned}$$

leading to the same approximate equation of motion as before. ☐

Exercise 2.28

(i) Examine the motion near the other boundary point for the system considered in the above example, using a similar approximation method.

(ii) Show that the equation of motion may be solved exactly by means of the substitution $x = \sin y$, and confirm that motion terminates on the right-hand boundary point for every initial position.

2.4.3 Motion in a circle

The phase space does not always occupy the whole real line. For some types of motion, typified by rotation about an axis, the phase space may be considered as a circle. In that case it is convenient to choose a coordinate θ in the range $[0, 2\pi]$ and to identify the ends of the range; that is, the point $\theta = 0$ is physically the same as the point $\theta = 2\pi$. Alternatively, one can think of θ as taking all real values, but with $\theta + 2n\pi$ representing the same point for every integer n.

The equation of motion for an autonomous system on a circle is

$$\dot{\theta} = v(\theta), \tag{2.17}$$

where $v(\theta)$ must be a periodic function of θ with period 2π, that is,

$$v(\theta) = v(\theta + 2\pi) \qquad \text{for all } \theta.$$

For instance, $v(\theta) = \sin\theta$ could be a periodic velocity function.

The simplest example of motion in a circle is motion with a constant velocity field, $v(\theta) = \omega$ with ω a constant. Then

$$\dot{\theta} = \omega, \qquad \text{so} \quad \theta(t) = \omega t + \theta_0,$$

which represents *uniform rotation* with period $T = 2\pi/\omega$, for $\omega \neq 0$.

Uniform rotation

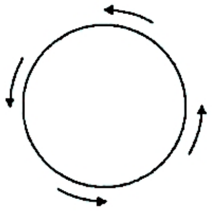


Figure 2.8 Phase diagram for uniform rotational motion, $v(\theta) = \omega$.

In fact, any circular motion with no fixed point is named a *rotation*.

Rotation

The *period* T of a rotation is the time taken for θ to increase by 2π, or in other words for the phase point to return to the same position on the circle. The period may be calculated by considering the motion of the point initially at $\theta = 0$; this gives

Period of a rotation

$$T = \int_0^T dt = \int_0^{2\pi} \frac{d\theta}{v(\theta)}. \tag{2.18}$$

In fact, the same result is obtained whatever the initial position, because of the periodicity of $v(\theta)$: see Exercise 2.30.

Exercise 2.29

Consider the system on the circle with equation of motion

$$\dot{\theta} = a + b\sin\theta \qquad (0 \le \theta \le 2\pi,\ a > 0,\ b > 0).$$

Find the fixed points and the invariant sets of the motion when $a < b$, $a = b$ and $a > b$. For what values of a and b is the motion a rotation? Find the period of rotational motion.

Exercise 2.30

Show that for every θ_0,

$$\int_{\theta_0}^{\theta_0+2\pi} \frac{d\theta}{v(\theta)} = \int_0^{2\pi} \frac{d\theta}{v(\theta)} = T.$$

Deduce that, for any solution $\theta(t)$ of the equation of motion of a rotation with period T,

$$\theta(t + T) = \theta(t) + 2\pi \qquad \text{for all } t.$$

2.5 Examples of first-order systems

We conclude our discussion of autonomous first-order systems with some examples taken from *MST204*, which we re-examine in the light of the concepts introduced in this unit. The examples considered are the models of the growth and exploitation of animal populations discussed in *Unit 3* of *MST204*, and the one-dimensional motion of a particle acted on by gravity and air resistance, which is treated in *MST204 Unit 4*.

2.5.1 Population growth

The dynamics of biological populations is a branch of ecology. The populations of insects, birds, fish and mammals are increased by birth and decreased by death. These processes depend upon many factors that are less well understood than the factors which influence the motion of levers, pulleys, projectiles, planets, atoms and molecules. Nevertheless, simplified models can help us to understand the way that these populations change with time.

When the population of a single species in some specified region of the world is sufficiently numerous it may be represented by a continuous real variable P. The values of P should really be whole numbers, of course, but we ignore that for the purposes of this model. The values must also be non-negative; this we can incorporate by restricting the phase space to be $[0, \infty]$, with a natural boundary at $P = 0$.

We shall use the notation of *MST204 Unit 3* as far as possible.

If the birth and death rates per individual of the population (also known as proportionate birth and death rates) depend only on the population size, and not on any other factors such as space and time, then the population can be modelled by a first-order autonomous system with the equation of motion

$$\dot{P} = v(P) = Pg(P), \tag{2.19}$$

where $g(P)$ is the difference between the proportionate birth and death rates, called the proportionate growth rate of the population.

The simplest assumption is that the proportionate birth rate B and the proportionate death rate C are independent of the population, in other words constants. Then the proportionate growth rate is also constant, and the equation of motion is

$$\dot{P} = (B - C)P.$$

This is a linear system, as discussed in Example 2.7 (page 15).

If we exclude the special case $B = C$, the natural boundary at $P = 0$ is the only fixed point. If $C > B$, the population decays exponentially towards zero, though it never reaches it; this is the case of a stable fixed point. If $B > C$, on the other hand, the population increases exponentially without bound; the fixed point at $P = 0$ is unstable.

In practice a population in a confined region cannot increase without bound, as there are limiting factors, such as competition for food and living space. The simple linear model is not a good one, therefore, and we must expect that the proportionate growth rate $g(P)$ is not constant.

It is not easy to specify the precise form of $g(P)$, but even so we can deduce some qualitative features of the behaviour of the system from straightforward assumptions about $g(P)$. In the first place, if $g(P)$ is always negative then $v(P)$ is also negative for all $P > 0$, and so the system will have a single fixed point at $P = 0$, to which all states with $P > 0$ tend. Thus the system in this case will behave qualitatively like a linear system with $C > B$, and the species is doomed to extinction.

If extinction is to be avoided, therefore, $g(P)$ must be positive for some values of P. Now the likely effect of the competition mentioned above is that the proportionate growth rate $g(P)$ will be a decreasing function of P. If we assume that $g(P)$ is positive for small values of P, at least, there are two possibilities to consider:

either $g(P)$ will tend to some positive limiting value as $P \to \infty$; or there will be a single value, say M, of P such that $g(M) = 0$. These alternatives are illustrated in Figure 2.9, together with the corresponding graphs of $v(P)$.

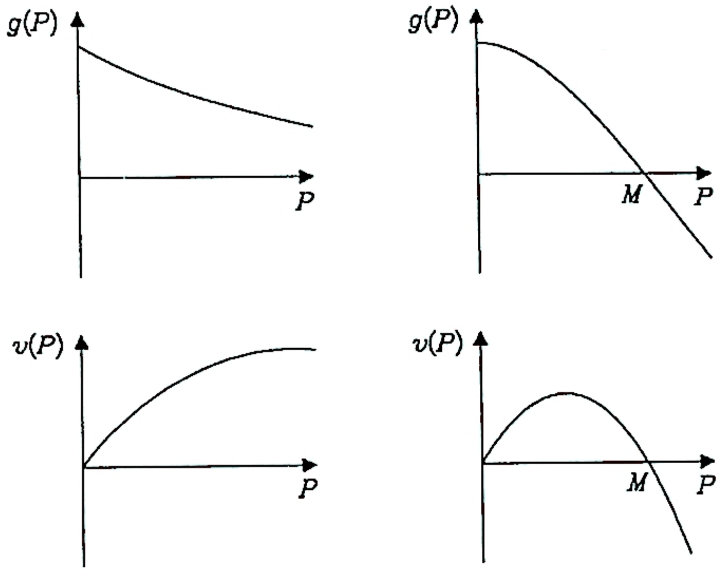


Figure 2.9 Graphs of $g(P)$ and $v(P)$ for decreasing $g(P)$.

In the first case $v(P)$ will be positive for all $P > 0$, the system will have a single unstable fixed point at $P = 0$, and the system will behave qualitatively like the linear case with $B > C$; in particular, the population will grow without bound. This possibility also fails to provide a satisfactory model and must therefore be discarded.

Note that a zero of $g(P)$ is also a zero of $v(P)$, so the second of the above possibilities introduces a new feature. In this case there are two fixed points, at 0 and M. Now

$$v'(P) = g(P) + Pg'(P). \tag{2.20}$$

The fixed point at 0 is therefore unstable, because

$$v'(0) = g(0),$$

which is positive by assumption. The fixed point at M must therefore be stable if it is simple, and this is clear from Equation 2.20, because

$$v'(M) = g(M) + Mg'(M) = Mg'(M),$$

and $g'(M) \leq 0$ since $g(P)$ is decreasing. The possibility that the fixed point at M is not simple will be ignored. Thus any proportionate growth rate function which is a decreasing function of P and is initially positive but eventually negative will lead to a model of the dynamics of the population with the following general features: there will be an unstable fixed point at $P = 0$, a stable fixed point at $P = M$ (where M is the single root of the equation $g(P) = 0$); if the population is in the interval $(0, M)$ it will increase and tend in the long term to M, while if it is in (M, ∞) it will decrease but still tend in the long term to M.

The simplest example of a decreasing function is a linear one. If we set $g(0) = a$, and denote the zero of $g(P)$ by M as above, then

$$g(P) = a\left(1 - \frac{P}{M}\right), \tag{2.21}$$

and the equation of motion has the form

$$\dot{P} = aP\left(1 - \frac{P}{M}\right), \tag{2.22}$$

which is the well-known logistic equation.

Exercise 2.31

(i) Confirm that the logistic equation has an unstable fixed point at $P = 0$ and a stable fixed point at $P = M$.

(ii) Without actually solving the equation, show that the solution with initial condition $P = P_0$ at $t = 0$ behaves approximately like

$$P(t) = P_0 e^{at}$$

for small P_0 and small t; and write down an exponential approximation to the solution for P_0 close to M.

(iii) In fact, the solution is

$$P(t) = \frac{M}{1 + (M/P_0 - 1)e^{-at}}.$$

Use this expression to confirm your answers to part (ii).

Exercise 2.32

Various modifications of the logistic equation have been suggested. One gives a population equation of the form

$$\dot{P} = aP\left\{1 - b(e^P - 1)\right\}, \qquad P \geq 0,\ a > 0,\ b > 0.$$

Describe qualitatively the change of population with time in this case.

We consider next the effect of exploitation of a population whose growth, if unexploited, is given by Equation 2.19 (page 24). The species in question might be a species of fish, for example, and the question of interest is to determine levels of catch for which the population is not driven to extinction by over-fishing. In general terms the effect of the inclusion of exploitation is to change Equation 2.19 to

$$\dot{P} = v(P) - c(P, t), \tag{2.23}$$

where $c(P, t)$ is the catch per unit time, that is, the rate of exploitation of the species. We shall assume that, if unexploited, the population would settle down to an equilibrium level M, and that $v(P) = Pg(P)$, where $g(P)$ is a decreasing function with $g(0) > 0$ and $g(M) = 0$. Then the graph of $v(P)$ is like the right-hand case in Figure 2.9.

There are two cases in which it is relatively easy to draw some qualitative conclusions about the effect of exploitation: that in which the rate of exploitation is a constant, and that in which it is proportional to the population size.

In the first case, $c(P, t)$ is a positive constant, say c, and Equation 2.23 becomes

$$\dot{P} = v(P) - c.$$

If c is not too large, the velocity function for this system looks like Figure 2.10.

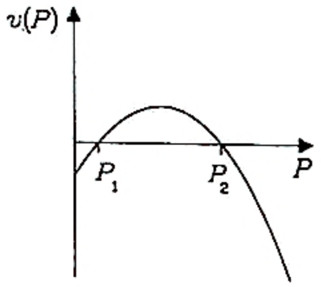


Figure 2.10 Graph of velocity function for exploitation at constant rate.

The effect of the exploitation term is to translate the graph of $v(P)$ downwards by c. The new system has fixed points at P_1 and P_2, with $0 < P_1 < P_2 < M$. The fixed point at P_1, like that at 0 for the unexploited population, is unstable; the fixed point at P_2, like that at M for the unexploited population, is stable.

Exercise 2.33

Describe the future of the exploited population if its current value P_0 satisfies

(i) $0 < P_0 < P_1$, (ii) $P_1 < P_0 < P_2$, (iii) $P_2 < P_0$.

Exercise 2.34

Let $v_{\max}$ be the maximum value of $v(P)$. Show that the conclusions of the previous discussion hold for $c < v_{\max}$. Explain what happens for $c > v_{\max}$. Find the critical value, $v_{\max}$, of c for the logistic equation.

When the rate of exploitation is proportional to population size, so that $c(P,t) = kP$, say (where k is a positive constant), the equation of change becomes

$$\dot{P} = v(P) - kP = P(g(P) - k).$$

Exercise 2.35

Analyse the possible different outcomes of exploitation in this case.

2.5.2 Motion under gravity and air resistance

Mechanical systems, even in one dimension, generally have order greater than one so do not fall within the scope of this unit. However, the equation of motion for a particle moving in one dimension under a force which depends on velocity but not position may be written

$$m\frac{dV}{dt} = F(V), \tag{2.24}$$

We use V for the particle's velocity in order to avoid confusion with the use of v elsewhere in the unit.

where $V(t)$ is the particle's velocity at time t, $F(V)$ is the force acting on it when its velocity is V, and m is its mass. This is a first-order differential equation for $V(t)$, so we may regard the system as a first-order one provided we take V to define its state. This is not the usual point of view in mechanics, where we normally want to know the time development of the position of the particle, not the velocity; however, there are interesting things to be said about Equation 2.24 in the light of the ideas developed in this unit, though one has to be careful in interpreting those ideas in this context.

In this case we shall resist calling the right-hand side of the equation of motion the velocity function, again to prevent confusion.

Exercise 2.36

What is the significance of the term 'equilibrium' in the present context?

For motion under gravity alone, in the usual approximation for motion near the Earth's surface, we have

$$F(V) = mg,$$

where g is the (constant) magnitude of the acceleration due to gravity, and the positive direction is taken as downward. This is a case of the simple type of system described in Example 2.4 (page 13).

More interesting situations occur when air resistance as well as gravity acts on the particle. Air resistance is a force whose magnitude depends on the speed, rather than the velocity, of the particle, and which one expects to increase in magnitude with increasing speed. The direction of the resistive force is always to oppose the motion, so that its direction is opposite to that of the velocity of the particle.

In order to avoid complications in this illustrative example we shall consider only the case of a particle *falling* through a resistive medium with velocity $V > 0$. A common assumption is that the resistance is proportional to the square of the speed, $F(V) = -kV^2$, with k a positive constant, so the equation of motion is

$$m\dot{V} = mg - kV^2, \qquad V > 0. \tag{2.25}$$

Exercise 2.37

(i) Provide a qualitative description of the motion of a particle governed by Equation 2.25. In particular, describe the motion when the initial velocity is $V(0) > \sqrt{mg/k}$ and when $V(0) < \sqrt{mg/k}$.

(ii) Solve the equation of motion and confirm your qualitative description.

Exercise 2.38

Consider the equation of motion for a particle falling under gravity and linear air resistance, $F(V) = -kV$. Provide a qualitative description, and confirm your predictions by solving the equation of motion.

Solutions to the Exercises in Unit 2

Solution 2.1

(i) $v(x,t) = \exp x$ is independent of t so the system is autonomous.

(ii) $v(x,t) = t$ depends only upon t so the system is non-autonomous.

(iii) In this case the velocity function $v(x,t)$ changes its form at $t = 0$, so it is dependent upon t (even though t does not appear explicitly in the definition) and the system is non-autonomous.

Solution 2.2

The graphical method described in *Unit 1*, Subsection 1.3.2, to find the solution of $\dot{x} = v(x,t)$ involves drawing sets of line segments in the (x,t)-plane such that the segment at the point (a,b) has gradient $v(a,b)$. If the velocity function is independent of t then all the direction lines at points with the same x-coordinate (but different t-coordinates) will be the same; in other words, the direction field will be unchanged by translation parallel to the t-axis.

Solution 2.3

Using separation of variables, provided $x \neq 0$ the differential equation can be written in the form

$$\int_{x_0}^{x} \frac{dx}{x} = b \int_{t_0}^{t} dt.$$

For $x_0 > 0$ we must have $x > 0$ for the left-hand integral to make sense; then

$$\int_{x_0}^{x} \frac{dx}{x} = \ln x - \ln x_0.$$

For $x_0 < 0$, on the other hand, we must have $x < 0$; then

$$\int_{x_0}^{x} \frac{dx}{x} = -\int_{x_0}^{x} \frac{dx}{-x} = \ln(-x) - \ln(-x_0).$$

But in either case

$$\int_{x_0}^{x} \frac{dx}{x} = \ln(x/x_0),$$

so

$$\ln(x/x_0) = b\,(t - t_0) \quad \Longrightarrow \quad x(t) = x_0 e^{b(t-t_0)}.$$

When $x_0 > 0$, $x(t) > 0$ for all t; when $x_0 < 0$, $x(t) < 0$ for all t. Finally, when $x_0 = 0$, the solution (which cannot be found by separation of variables) is the constant function $x(t) = 0$.

For later use it is important to note how the nature of the solution depends on the sign of b. If $b > 0$, then for $x_0 > 0$, $x(t)$ increases steadily and more and more quickly when t increases, and $x(t) \to +\infty$ as $t \to +\infty$; while for $x_0 < 0$, $x(t)$ decreases and $x(t) \to -\infty$ as $t \to +\infty$. If $b < 0$, on the other hand, $x(t) \to 0$ as $t \to +\infty$, whether $_0 > 0$ or $x_0 < 0$.

Solution 2.4

If the autonomous system is $\dot{x} = v(x)$ and $x(t) = c$ is a solution, then $\dot{x} = 0$, so it is necessary that $v(c) = 0$. Conversely, if $v(c) = 0$ then $\dot{x} = 0$ when $x = c$, and so the equation of motion has the constant solution $x(t) = c$.

Solution 2.5

For this equation we can use the method of separation of variables to write, for $x_0 > 0$ and $x > 0$,

$$\int_{x_0}^{x} \frac{dx}{x} = \int_{t_0}^{t} dt\, t \qquad \text{or} \qquad \ln(x/x_0) = \tfrac{1}{2}(t^2 - t_0^2).$$

Hence

$$x(t) = x_0 \exp\left\{\tfrac{1}{2}(t^2 - t_0^2)\right\},$$

which depends upon both t and t_0 and not just $t - t_0$. The same equation gives the solution for $x_0 \leq 0$.

Solution 2.6

Set $\tau = t_1 - t_0$, for convenience, and let $\tilde{x}(t) = x(t - \tau)$. Then $\tilde{x}(t)$ is certainly a solution of the differential equation, exactly as in the text. Moreover,

$$\tilde{x}(t_1) = x(t_1 - \tau) = x(t_0) = x_0,$$

and so $\tilde{x}(t)$ satisfies the given initial condition. It is thus the required solution.

Solution 2.7

Again, $\tilde{x}(t) = x(s + t)$ is a solution of the differential equation, and $\tilde{x}(0) = x(s)$, so it is the required solution.

This result may be interpreted as follows: for an autonomous first-order system, two phase points which pass through the same initial point a time s apart follow the same solution, but always separated by the same time s.

Solution 2.8

The general solution of $\dot{x} = v(t)$ is $x(t) = \int dt\, v(t)$, so invariance under translation parallel to the x-axis is equivalent to the fact that indefinite integrals are defined only to within additive constants.

Solution 2.9

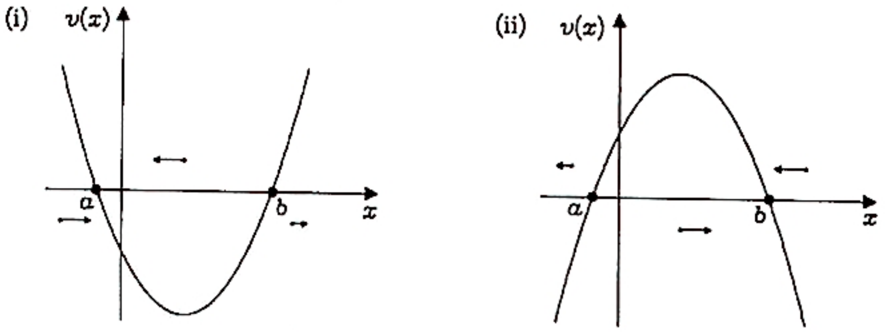


Figure S2.1 Phase diagrams for
(i) $v(x) = (x - a)(x - b)$, (ii) $v(x) = (a - x)(x - b)$, with $a < b$.

The fixed points in each case are a and b, and the elementary invariant sets are these points and the intervals $(-\infty, a)$, (a, b) and (b, ∞). The phase diagrams for the two functions are the same in every respect but one: the directions of the arrows are opposite in the two cases. This is simply a consequence of the fact that the velocity function of part (ii) is minus that of part (i).

Solution 2.10

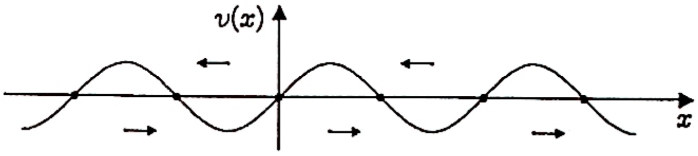


Figure S2.2 Phase diagram for $v(x) = \sin x$.

The fixed points are the points where $\sin x = 0$, namely $x = n\pi$ for every integer n. The elementary invariant sets are these points and the intervals $(n\pi, (n+1)\pi)$. In this case there are infinitely many elementary invariant intervals, but each is finite in length.

Solution 2.11

A fixed point, x_f, is invariant under the flow so

$$x_f = \phi(x_f, t) \qquad \text{or, equivalently,} \qquad x_f = \phi_t(x_f).$$

Solution 2.12

It was shown in Exercise 2.7 that for any solution $x(t)$ and any s, $x(s+t)$ is also a solution, with initial condition $x = x(s)$ when $t = 0$. In terms of the notation for flows, this means that

$$\phi(x, t+s) = \phi(\phi(x,t), s) \qquad \text{or} \qquad \phi_{t+s}(x) = \phi_s(\phi_t(x)).$$

Similarly, we have

$$\phi_{s+t}(x) = \phi_{t+s}(x) = \phi_t(\phi_s(x))$$

on reversing the roles of s and t. Notice that one consequence of this result is that

$$\phi_s(\phi_t(x)) = \phi_t(\phi_s(x))$$

(simply because $s + t = t + s$); that is, ϕ_s and ϕ_t commute.

When $\phi_t(x) = x + at$ we have

$$\phi_s(\phi_t(x)) = \phi_s(x + at) = (x + at) + as = x + a(s+t) = \phi_{s+t}(x).$$

When $\phi_t(x) = e^{bt}x$, on the other hand,

$$\phi_s(\phi_t(x)) = \phi_s(e^{bt}x) = e^{bs}(e^{bt}x) = (e^{bs}e^{bt})x = e^{b(s+t)}x = \phi_{s+t}(x).$$

Solution 2.13

From the result of Exercise 2.12,

$$\phi_{-t} \circ \phi_t = \phi_t \circ \phi_{-t} = \phi_0,$$

which is the identity transformation; thus ϕ_{-t} is the inverse of ϕ_t.

As one might expect for an autonomous system, reversing the direction of time has the effect of bringing you back to where you started.

Solution 2.14

This follows directly from the results in the text if ξ is replaced by the appropriate expression in x: we have $\xi(t) = \xi_0 e^{bt}$, $\xi = x - x_f$, whence $\xi_0 = x_0 - x_f$, and so

$$x(t) - x_f = (x_0 - x_f)e^{bt}, \qquad \text{where} \quad x_0 = x(0),$$

as required.

Alternatively, one can easily solve the equation of motion without making a substitution. Using the integrating factor method, the equation can be written in the form

$$\frac{dx}{dt} - bx = a \quad \text{or} \quad \frac{d}{dt}\left(xe^{-bt}\right) = ae^{-bt}$$

and has the solution

$$x(t) = -\frac{a}{b} + \left(x_0 + \frac{a}{b}\right)e^{bt} = x_f + (x_0 - x_f)e^{bt}.$$

If $x_0 = x_f$ then $x(t) = x_f$ for all times. If $b < 0$ then $x(t) \to x_f$ as $t \to \infty$ for any x_0. But if $b > 0$ then $|x(t)| \to \infty$ as $t \to \infty$. These conclusions become particularly evident if we write the solution in the form $x(t) - x_f = (x_0 - x_f)e^{bt}$.

Notice that as $t \to -\infty$ then $|x(t)| \to \infty$ for $b < 0$ and $x(t) \to x_f$ for $b > 0$. So reversing the direction of time has the effect of reversing the conclusions about the behaviour of the solutions.

Solution 2.15

The fixed points are 0, ±1. The derivative of the velocity function is given by

$$v'(x) = 3x^2 - 1,$$

and so

$$v'(0) = -1, \qquad v'(\pm 1) = 2.$$

Thus all the fixed points are simple; 0 is stable, while ±1 are unstable.

Solution 2.16

In each case the fixed points are clearly a and b.

(i) In this case

$$v'(x) = (x - b) + (x - a) = 2x - (a + b),$$

so

$$v'(a) = a - b < 0, \qquad v'(b) = b - a > 0.$$

Thus both fixed points are simple, a is stable and b is unstable.

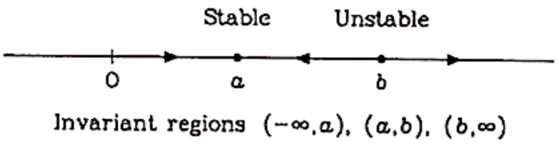


Figure S2.3

(ii) In this case

$$v'(x) = -(x - b) + (a - x) = a + b - 2x,$$

so

$$v'(a) = b - a > 0, \qquad v'(b) = a - b < 0.$$

As before, both fixed points are simple; but this time a is unstable and b is stable.

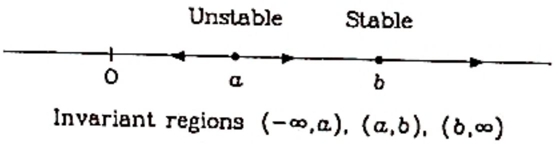


Figure S2.4

These conclusions are clear also from the solution to Exercise 2.9 (page 30). Notice that the effect of changing the sign of the velocity function is to change stable fixed points to unstable ones, and vice versa.

Solution 2.17

For a first-order system with exactly two simple fixed points it is not possible for the fixed points to be of the same type. This is clear because the arrow in the finite elementary invariant interval between the two fixed points must point from one fixed point to the other. Thus one fixed point must be stable and the other unstable. There are two essentially different possibilities, one in which the stable fixed point is to the left of the unstable one, and one in which their positions are reversed; each is the time reversal of the other. The velocity functions treated in the previous exercise are examples of the two possible cases; their phase diagrams, which are models for the general case, are shown in more detail in the solution to Exercise 2.9 (page 30).

Solution 2.18

The key point here is that the substitution of $-t$ for t in the equation of motion of an autonomous system changes it from $\dot{x} = v(x)$ to $\dot{x} = -v(x)$; that is, changing the sign of the velocity function is equivalent to reversing the direction of time. But reversing the direction of time interchanges stable and unstable fixed points. Thus the system with equation of motion

$$\dot{x} = -v(x)$$

has the required property.

Solution 2.19

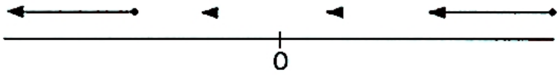


Figure S2.5 Phase diagram for the motion $\dot{x} = cx^2$, $c < 0$.

Points to the left of the fixed point 0 leave it, points to the right of it approach it. Once again, the effect of changing the sign of the velocity function is to change the directions of the arrows in the phase diagram; but in this case this does not affect the stability of the fixed point.

Solution 2.20

It is worth remarking to begin with that the velocity function has a local extremum at x_f, which is a minimum if $v''(x_f) > 0$ and a maximum if $v''(x_f) < 0$. The shape of the velocity function near the fixed point x_f is thus similar to the shape of cx^2 near zero, where c has the appropriate sign. It is to be expected that the nature of the fixed point x_f will be the same as that of the fixed point 0 of the velocity function cx^2.

This conclusion can be confirmed by making a Taylor expansion. The second-order Taylor expansion of $v(x)$ about x_f is

$$v(x) = \tfrac{1}{2}(x - x_f)^2\, v''(x_f) + O\big((x - x_f)^3\big)\,.$$

Thus for points near x_f the equation of motion may be approximated by

$$\dot{x} = \tfrac{1}{2}(x - x_f)^2\, v''(x_f),$$

which is similar to the equation of motion for the velocity function cx^2 with $c = \frac{1}{2}v''(x_f)$, and may be made identical to it by means of the substitution $\xi = x - x_f$. The fixed point x_f is neither stable nor unstable; motion on either side of it is to the right for $v''(x_f) > 0$, to the left for $v''(x_f) < 0$.

Solution 2.21

Motion near a quadratic fixed point at 0 is approximated by the equation of motion $\dot{x} = cx^2$. This can be solved by separation of variables, which gives

$$\int_{x_0}^{x} \frac{dx}{x^2} = c\int_0^t dt = ct,$$

where x must have the same sign as x_0, which must not be zero. Direct integration now gives

$$\frac{1}{x_0} - \frac{1}{x} = ct \qquad \text{or} \qquad x(t) = \frac{x_0}{1 - cx_0 t},$$

which is certainly not exponential. The more general case of a quadratic fixed point at x_f may be treated by using the substitution $\xi = x - x_f$ in the usual way.

Notice that even though the motion is not exponential, it remains true, as for a simple fixed point, that for $x_0 \neq 0$ the fixed point cannot be reached in a finite time.

The solution of this equation of motion is considered further in Exercise 2.26.

Solution 2.22

The velocity function $v(x) = dx^3$ has a single fixed point at $x = 0$, which is clearly not simple. However, for $d > 0$ we have $v(x) > 0$ for $x > 0$ and $v(x) < 0$ for $x < 0$, which means that states near 0 move away from it as time increases, so the fixed point is unstable; while if $d < 0$ we have $v(x) < 0$ for $x > 0$ and $v(x) > 0$ for $x < 0$, which means that states near 0 move towards it as time increases, so the fixed point is stable.

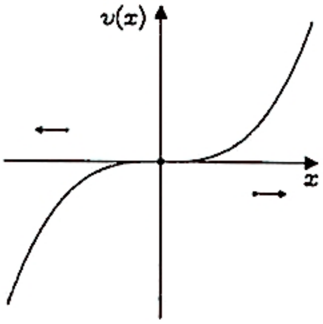


Figure S2.6 Phase diagram for the motion $\dot{x} = dx^3$, $d > 0$.

Separation of variables in the equation of motion gives, for $x_0 \neq 0$,

$$\int_{x_0}^{x} \frac{dx}{x^3} = d\int_0^t dt = td,$$

where x must have the same sign as x_0. Thus

$$\frac{1}{2x_0^2} - \frac{1}{2x^2} = td \qquad \text{or} \qquad x(t) = \frac{x_0}{\sqrt{1 - 2dx_0^2 t}}, \tag{S2.1}$$

which again is not exponential. Even so, when $d < 0$ we have $1 - 2dx_0^2 t > 0$ for all $t \geq 0$, and $x(t) \to 0$ as $t \to \infty$. However, there is no finite value of t such that $x(t) = 0$.

Solution 2.23

It remains true that adjacent fixed points cannot both be stable or both be unstable, for the reason given before. But now one or both of the fixed points can be neither stable nor unstable. This means that any assignment of directions to the elementary invariant intervals represents a possible flow structure for the system. There are three elementary invariant intervals, and two possible directions for each, which means that there are eight essentially different phase diagrams. Four of them are illustrated in Figure S2.7 below, with the graphs of representative velocity functions;

the others are obtained by reversing the arrows on these, which is equivalent to reversing the direction of time, or reflecting the graph of the velocity function in the x-axis.

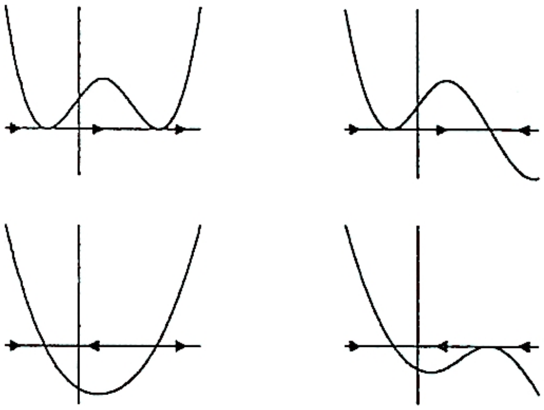


Figure S2.7 Phase diagrams for systems with two fixed points.

Solution 2.24

In the first case the fixed points are at the roots of

$$\overline{v}(x) = bx + \varepsilon = 0.$$

There is thus just one fixed point, at $x = -\varepsilon/b = x_f$. We have $\overline{v}'(x) = b$ for all x, and in particular for $x = x_f$; thus the nature of the fixed point is the same for every ε.

In the second case the fixed points are at the real roots, if there are any, of

$$cx^2 + \varepsilon = 0 \qquad \text{or} \qquad x^2 = -\varepsilon/c.$$

$\varepsilon/c > 0$ there are no real solutions to this equation, hence no fixed points.

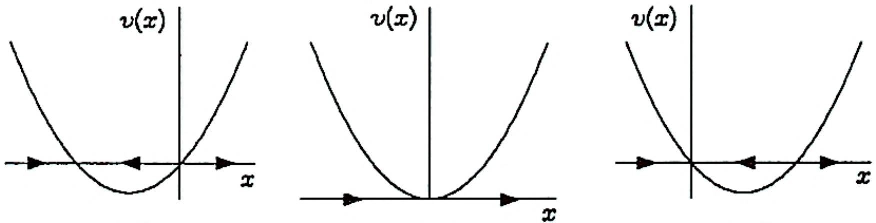
$\varepsilon = 0$ there is one fixed point, which is neither stable nor unstable.

$\varepsilon/c < 0$ there are two roots, at $x = x_\pm = \pm\sqrt{-\varepsilon/c}$. Now $\overline{v}'(x) = 2cx$, therefore (x_+) has the same sign as c, $\overline{v}'(x_-)$ the opposite sign to c, so we have one stable and one unstable fixed point.

Thus as ε/c decreases from positive values through 0 to negative values, the number and nature of the fixed points of the system with velocity function $\overline{v}(x)$ changes dramatically, from no fixed points to two fixed points of opposite type; $\varepsilon = 0$ is the critical value, at which the twin fixed points are born.

Solution 2.25

Phase diagrams for the velocity function $\overline{v}(x) = cx^2 + \varepsilon x$ for $c > 0$ and $\varepsilon > 0$, $\varepsilon = 0$ and $\varepsilon < 0$ are shown in Figure S2.8.



$\varepsilon > 0$ $\varepsilon = 0$ $\varepsilon < 0$

Figure S2.8 Graphs of the function $\overline{v}(x) = cx^2 + \varepsilon x$, and corresponding phase flows, for $c > 0$ and for various values of ε.

Note that $\overline{v}'(0) = \varepsilon$, so the nature of the fixed point at 0 depends on ε but not on c. For $\varepsilon < 0$ the fixed point at $x = 0$ is stable, for $\varepsilon > 0$ it is unstable and for $\varepsilon = 0$ it neither.

In this system, although the position of the fixed point under examination does not change, its nature depends upon ε, and changes as ε passes through the critical value zero.

Solution 2.26

We derived the solution to this system in Exercise 2.21; it is given by

$$x(t) = \frac{x_0}{1 - cx_0t},$$

where $x_0 \neq 0$ and x has the same sign as x_0. However, this solution is not defined for $t = (cx_0)^{-1}$. The function given above is well-defined for $t > (cx_0)^{-1}$ and for $t < (cx_0)^{-1}$; we require the solution to be defined when $t = 0$, and must therefore choose the infinite interval which includes this value of t. When $c > 0$ and $x_0 > 0$ the required range for t is $t < (cx_0)^{-1}$. The solution is thus

$$x(t) = \frac{x_0}{1 - cx_0t}, \qquad t < (cx_0)^{-1}.$$

Solution 2.27

Separation of variables gives

$$\int_{x_0}^{x} dx\, x^{-n} = t \qquad \text{or} \qquad x^{1-n} - x_0^{1-n} = (1-n)t \qquad (n \geq 2).$$

This leads to the solution

$$x(t) = \frac{x_0}{\left(1 - (n-1)x_0^{n-1}t\right)^{1/(n-1)}}.$$

Notice that this equation reduces to the solution of the previous exercise when $n = 2$ and $c = 1$ and to Equation S2.1 (page 34) when $n = 3$ and $d = 1$. Here $x_0 \neq 0$, and $x(t)$ has the same sign as x_0. The function above is not defined at $t = x_0^{1-n}/(n-1)$, and so the motion terminates there. If n is even then $x_0^{1-n}/(n-1) > 0$ for $x_0 > 0$, and so the motion is defined for $t < x_0^{1-n}/(n-1)$, while $x_0^{1-n}/(n-1) < 0$ for $x_0 < 0$ and the motion is defined for $t > x_0^{1-n}/(n-1)$. For n odd, $x_0^{1-n}/(n-1) > 0$ whatever the sign of x_0, and the motion is defined for $t < x_0^{1-n}/(n-1)$ in either case.

Solution 2.28

(i) Using the substitution method, near $x = -1$ we put $x = -1 + \xi$, or $\xi = 1 + x$, to give

$$\dot{\xi} = \sqrt{2\xi} + O\left(\xi^{3/2}\right),$$

which is similar to Equation 2.14 studied in the text (page 21), but differs from it in sign. The solution is therefore

$$\xi(t) = \left(\sqrt{\xi_0} + t/\sqrt{2}\right)^2,$$

whence

$$x(t) = -1 + \left(\sqrt{1 + x_0} + t/\sqrt{2}\right)^2.$$

In this case $x(t)$ is increasing away from the boundary point at $x = -1$ as time increases; the boundary point was left at a time $t = -\sqrt{2(1 + x_0)}$; for earlier times the motion is undefined.

(ii) In this instance we can solve the equation of motion exactly by defining $x = \sin y$, $-\pi/2 \leq y \leq \pi/2$, so $\dot{x} = \dot{y}\cos y$, and since $1 - \sin^2 y = \cos^2 y$, the equation of motion is

$$\dot{y} = 1 \qquad \text{with solution} \quad y = y_0 + t.$$

Thus

$$x(t) = \sin\left(t + \sin^{-1} x_0\right).$$

On putting $x(t) = 1$ we find the critical time, at which the motion terminates, to be

$$t_c(x_0) = \tfrac{1}{2}\pi - \sin^{-1} x_0.$$

Solution 2.29

The fixed points are solutions of

$$\sin\theta = -a/b.$$

$a < b$ then there are two fixed points, at $\theta = \tfrac{3}{2}\pi \pm \alpha$ where $\cos\alpha = a/b$, $\leq \alpha \leq \tfrac{1}{2}\pi$, as can be seen from the graph of $v(\theta) = a + b\sin\theta$. This also shows that the fixed point at $\tfrac{3}{2}\pi - \alpha$ is stable and that at $\tfrac{3}{2}\pi + \alpha$ is unstable. There are two elementary invariant sets (apart from the fixed points themselves), namely the arcs into which the circle is divided when the two fixed points are removed.

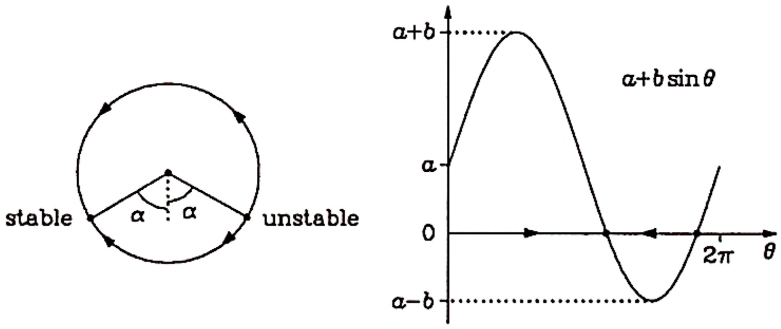


Figure S2.9

When $a = b$ there is a single fixed point, at $\theta = \tfrac{3}{2}\pi$, which is neither stable nor unstable. The elementary invariant sets are the fixed point, and the whole of the remainder of the circle.

For $a > b$ there are no fixed points and the motion is purely rotational. The only Invariant set is the whole circle. The period is given by

$$T = \int_0^{2\pi} \frac{d\theta}{a + b\sin\theta} = \int_{-\pi}^{\pi} \frac{d\theta}{a + b\sin\theta}.$$

This integral can be evaluated using a variety of techniques. One method is to first substitute $t = \tan\tfrac{1}{2}\theta$ and use the relations

$$\sec^2\tfrac{1}{2}\theta = 1 + t^2,$$

$$\cos\theta = \cos^2\tfrac{1}{2}\theta - \sin^2\tfrac{1}{2}\theta = \frac{1-t^2}{1+t^2},$$

$$\sin\theta = 2\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta = \frac{2t}{1+t^2},$$

Whence

$$\frac{2\,dt}{1+t^2} = d\theta.$$

This gives T in the form

$$T = \frac{2}{a}\int_{-\infty}^{\infty} \frac{dt}{1 + 2bt/a + t^2} = \frac{2}{a}\int_{-\infty}^{\infty} \frac{dt}{(t + b/a)^2 + (1 - b^2/a^2)}.$$

Now put

$$t + b/a = \sqrt{(1 - b^2/a^2)}\tan\phi, \qquad \text{so} \quad dt = \sqrt{(1 - b^2/a^2)}\sec^2\phi\, d\phi,$$

and use the relation $1 + \tan^2\phi = \sec^2\phi$ to obtain

$$T = \frac{2\pi}{\sqrt{a^2 - b^2}}.$$

Note that as $b \to a$, $T \to \infty$.

Solution 2.30

Suppose first that $0 < \theta_0 < 2\pi$. Then

$$\begin{aligned}\int_0^{2\pi} \frac{d\theta}{v(\theta)} &= \int_0^{\theta_0} \frac{d\theta}{v(\theta)} + \int_{\theta_0}^{2\pi} \frac{d\theta}{v(\theta)} \\ &= \int_{2\pi}^{\theta_0+2\pi} \frac{d\theta}{v(\theta)} + \int_{\theta_0}^{2\pi} \frac{d\theta}{v(\theta)} \qquad \text{(since } v(\theta) \text{ is periodic)} \\ &= \int_{\theta_0}^{\theta_0+2\pi} \frac{d\theta}{v(\theta)}.\end{aligned}$$

If $2n\pi \le \theta_0 \le 2(n+1)\pi$ for some integer $n \neq 0$ then a similar argument shows that

$$\int_{\theta_0}^{\theta_0+2\pi} \frac{d\theta}{v(\theta)} = \int_{2n\pi}^{2(n+1)\pi} \frac{d\theta}{v(\theta)},$$

which is equal to $\int_0^{2\pi} d\theta/v(\theta)$, again by the periodicity of $v(\theta)$.

If the solution of the equation of motion with initial condition $\theta = \theta_0$ when $t = 0$ is $\theta(t)$ then

$$\int_{\theta_0}^{\theta(t)} \frac{d\theta}{v(\theta)} = t.$$

Thus

$$\begin{aligned}\int_{\theta_0}^{\theta(t+T)} \frac{d\theta}{v(\theta)} &= t + T \\ &= \int_{\theta_0}^{\theta(t)} \frac{d\theta}{v(\theta)} + \int_{\theta(t)}^{\theta(t)+2\pi} \frac{d\theta}{v(\theta)} \\ &= \int_{\theta_0}^{\theta(t)+2\pi} \frac{d\theta}{v(\theta)}.\end{aligned}$$

It follows that

$$\theta(t+T) = \theta(t) + 2\pi \qquad \text{for all } t.$$

Solution 2.31

(i) For the logistic equation, $v(P) = aP(1 - P/M)$, so $v(P) = 0$ at $P = 0$ and $P = M$. Furthermore, $v'(P) = a(1 - P/M) - aP/M$, so $v'(0) = a$ and $v'(M) = -a$. There is thus an unstable fixed point at 0 and a stable one at M.

(ii) In the linear approximation for the motion near a fixed point x_f of a system with velocity function $v(x)$, the approximate solution is

$$x(t) = x_f + (x_0 - x_f)e^{bt},$$

where $b = v'(x_f)$ (see Equation 2.12, page 16). For the fixed point at 0 of the logistic equation we have $v'(0) = a$, so this gives

$$P(t) = P_0 e^{at};$$

for the fixed point at M, $v'(M) = -a$, so the approximation is

$$P(t) = M + (P_0 - M)e^{-at}.$$

(iii) For the case of small P_0, multiply the solution top and bottom by $(P_0/M)e^{at}$ to obtain

$$P(t) = \frac{P_0 e^{at}}{1 + P_0/M(e^{at} - 1)};$$

then for small P_0 and t the denominator is small, giving the approximation

$$P(t) = P_0 e^{at}$$

as required. For P_0 close to M, on the other hand, and for $t \geq 0$, we can treat the factor $(M/P_0 - 1)e^{-at}$ in the denominator as small and write

$$\begin{aligned} P(t) &= M\left(1 - (M/P_0 - 1)e^{-at} + O((M - P_0)^2)\right) \\ &= M - P_0(M/P_0 - 1)e^{-at} + O((M - P_0)^2), \end{aligned}$$

which leads to the approximation

$$P(t) = M + (P_0 - M)e^{-at}.$$

The substitution of a P_0 for an M in the penultimate step, which does not affect the accuracy of the approximation because it only introduces an additional term of order $(M - P_0)^2$, is necessary to ensure that the approximate solution also satisfies the initial condition $P = P_0$ when $t = 0$.

Solution 2.32

The solution of this differential equation cannot be expressed simply in terms of known functions. But we can obtain a qualitative picture of the motion by examining the behaviour of the velocity function

$$v(P) = aP\left\{1 - b\left(e^P - 1\right)\right\}, \qquad P \geq 0,$$

and the corresponding proportionate growth rate function

$$g(P) = a\left\{1 - b\left(e^P - 1\right)\right\}.$$

Now $g(P)$ is a decreasing function of P; $g(0) = a > 0$, while $g(P) \to -\infty$ as $P \to \infty$, so there is a single positive value M of P for which $g(M) = 0$. Thus a population governed by this equation behaves qualitatively in the same way as one which follows the logistic equation.

We can find M explicitly:

$$be^M = 1 + b, \qquad \text{so} \quad M = \ln(1 + 1/b).$$

Furthermore, $g'(P) = -abe^P$, so

$$\begin{aligned} v'(M) = Mg'(M) &= -ab(1 + 1/b)\ln(1 + 1/b) \\ &= -a(1 + b)\ln(1 + 1/b) = -c, \quad \text{say}, \end{aligned}$$

which is certainly negative, confirming that M is a stable fixed point. It follows that for small initial populations, $P(t)$ initially grows approximately exponentially as

$$P(t) = P_0 e^{at},$$

just as in the logistic case; while for populations initially close to M, we have approximately

$$P(t) = M + (P_0 - M)e^{-ct},$$

with M and c as given above.

Solution 2.33

The fixed point at P_1 is unstable, while that at P_2 is stable. Thus

(i) if $0 < P_0 < P_1$, the population will dwindle to zero;

(ii) if $P_1 < P_0 < P_2$, the population will increase and tend towards P_2;

(iii) if $P_2 < P_0$, the population will decrease and tend towards P_2.

Solution 2.34

If the maximum value of $v(P)$ is $v_{\max}$ then the maximum value of $v(P) - c$ is $v_{\max} - c$. The previous conclusions hold only if $v(P) - c$ is positive somewhere, which requires that its maximum value should be positive, in other words that $v_{\max} > c$. If $v_{\max} < c$, then the velocity function for the exploited population will be negative always, which means that the population will decrease to the natural boundary $P = 0$, which it will reach in a finite time because $v(0) - c \neq 0$. If $v_{\max} = c$, then there is a single fixed point at which the graph of the velocity function touches the P-axis from below. This will be neither stable nor unstable; but in terms of the problem it is effectively unstable in the sense that the slightest amount of over-exploitation, with the population at this equilibrium level, will drive the species to extinction. Moreover, in this case the model is highly sensitive to changes in the values of relevant parameters, the shape of the growth curve and so on; one consequence of error is that the actual population could be destined for extinction.

The maximum value of $v(P)$ for the logistic equation occurs where $P = \frac{1}{2}M$, and $v_{\max} = \frac{1}{4}aM$. This is the critical value of c. It is called the maximum sustainable yield, because it is the maximum rate of exploitation for which the population can avoid extinction; but, as we have remarked above, it is questionable whether it is actually sustainable.

Solution 2.35

The crucial question in this case is whether $k > g(0)$ or $k < g(0)$. In the former case, since $g(P)$ is a decreasing function, the corresponding velocity function will always be negative, and the species is doomed to extinction. In the latter, there will be a single positive value M' of P for which $g(M') - k = 0$; this will be a stable fixed point, so the population will behave qualitatively as before, though of course $M' < M$.

Note that, with the population at its equilibrium level M', the rate of yield of exploitation is kM'. To find the maximum yield (assuming that the population will settle quickly to its equilibrium when exploitation begins) we have to find the maximum value of kM', thinking of this as a function of k. Now M' depends on k, so the answer is not immediately obvious. However, M' is determined by the equation $g(M') = k$, which means that the maximum rate of yield is equal to the maximum value of $M'g(M')$, that is, of the velocity function. Thus the maximum sustainable yield in this case is $v_{\max}$, as before. This time, however, when the rate of exploitation is fixed to give the maximum sustainable yield, the velocity function will have a simple stable fixed point, and the system will be insensitive to perturbations. Thus the fact that the model used to set the target catch levels is of necessity somewhat indeterminate will not carry the danger of extinction of the species.

Solution 2.36

The point of this question is that, in mechanics, the term 'equilibrium' usually implies that the body concerned is stationary. In the present context this is not so: an equilibrium state is one in which the particle moves with constant velocity, not necessarily zero.

Solution 2.37

(i) The fixed points are at $mg = kV^2$, and since $V > 0$ there is only one fixed point, $V = \sqrt{mg/k}$. If the initial velocity is $V(0) < \sqrt{mg/k}$, then the speed of the falling particle will steadily increase towards this limit but will never reach it in a finite time. If $V(0) > \sqrt{mg/k}$, then the particle's speed will decrease towards this limit.

(ii) The equation of motion is separable, so that by writing $V_0 = V(0)$ and $\alpha^2 = mg/k$, it can be expressed in the form of the integral

$$\begin{aligned} t &= \frac{m}{k}\int_{V_0}^{V} \frac{dV}{\alpha^2 - V^2} \\ &= \frac{1}{2}\sqrt{\frac{m}{kg}}\int_{V_0}^{V} dV \left(\frac{1}{\alpha - V} + \frac{1}{\alpha + V}\right). \end{aligned}$$

This may be integrated directly to give

$$t = \frac{1}{2}\sqrt{\frac{m}{kg}}\left[\ln\left|\frac{\alpha + V}{\alpha - V}\right|\right]_{V_0}^{V},$$

which can be rearranged into the form

$$\frac{\alpha + V}{\alpha - V} = \frac{\alpha + V_0}{\alpha - V_0}\exp(2gt/\alpha),$$

or, alternatively,

$$V(t) = \alpha\frac{(\alpha + V_0)e^{2gt/\alpha} + V_0 - \alpha}{(\alpha + V_0)e^{2gt/\alpha} + \alpha - V_0}.$$

From this last equation we see that for large t the exponential term dominates so $V(t) \to \alpha$ as $t \to \infty$.

Solution 2.38

For linear air resistance the equation of motion is $m\dot{V} = mg - kV$ and the terminal velocity is given by

$$V_{\mathrm{T}} = mg/k.$$

The equation of motion is of the linear type discussed in Example 2.7 and Exercise 2.14 (pages 15 and 16). The solution quoted in the latter, transcribed to the present case, gives

$$V(t) = V_{\mathrm{T}} + (V_0 - V_{\mathrm{T}})e^{-kt/m},$$

from which it is clear that $V(t) \to V_{\mathrm{T}}$ as $t \to \infty$.

UNIT 3

SECOND-ORDER AUTONOMOUS SYSTEMS

3.1 Introduction

Most systems are more complicated than the first-order systems described in *Unit 2*. With the exception of very special limiting cases, all Newtonian mechanical systems are of higher order than the first; most models for population growth involve several interacting species, and many complex chemical reactions, electrical circuits and economic models provide examples of higher-order systems. In this unit we shall be concerned with systems of order two.

An example of a special case of one-dimensional Newtonian motion treated as a first-order system is given in *Unit 2* Subsection 2.5.2.

For a second-order system two independent variables are needed to locate a given state in phase space. The phase space could be a plane, or part of a plane, the surface of a cylinder or sphere, or a more complicated surface. The simplest examples are planes, so in this unit we shall deal mainly with those.

The state variables for a second-order system in the plane can be thought of as Cartesian coordinates; we shall denote them by (x, y) in the usual way. The equations of motion for a second-order system then take the form

$$\frac{dx}{dt} = \dot{x} = v_x(x, y, t), \qquad \frac{dy}{dt} = \dot{y} = v_y(x, y, t),$$

which are two coupled first-order differential equations for the variables x and y in terms of the time t. A solution $(x(t), y(t))$ of the equations of motion determines a curve in phase space; any possible motion of the system corresponds to the tracing out of such a phase curve.

The main ingredient of this unit is the study of second-order systems and their phase curves by graphical methods. A very important aspect of this, just as in the first-order case, is the classification of the fixed points or equilibrium points of the system, and this will take up much of the unit. Systems whose equations of motion are linear, have constant coefficients and are homogeneous, which we shall call linear systems for short, provide models of the possible behaviour of an arbitrary system near a fixed point which cover most cases of interest. This follows the pattern set out in *Unit 2* for first-order systems. There we showed that a linear system has a single fixed point, which is simple, and can be either stable or unstable. For a second-order linear system, provided that the matrix of coefficients is non-singular, there is also just one fixed point, and it is simple; however, the transition from first- to second-order systems leads to a much greater variety of possible types of fixed point. There are, in fact, ten different types of simple fixed point of a second-order linear system – in the sense that they exhibit qualitatively different types of motion – which correspond to ten different classes of coefficient matrix, as determined by their eigenvalues and eigenvectors.

This unit rests heavily on the discussions of eigenvalues and eigenvectors in *Unit 1* Subsection 1.4.4, and of systems of differential equations in *Unit 1* Section 1.5, which in turn are based on *Units 21* and *22* of *MST204*.

3.2 Second-order systems: basic theory

The functions $v_x(x, y, t)$ and $v_y(x, y, t)$ which appear on the right-hand sides of the equations of motion

$$\dot{x} = v_x(x, y, t), \qquad \dot{y} = v_y(x, y, t) \tag{3.1}$$

are the components of the *velocity field* of the system, whose value for particular (x, y) and t is the phase velocity. We shall often use vector notation, writing $\mathbf{x}$ for the vector $x\,\mathbf{i} + y\,\mathbf{j}$, where $\mathbf{i}$ and $\mathbf{j}$ are the Cartesian unit vectors, or equally for (x, y), and $\mathbf{v}$ for the velocity vector field $v_x\,\mathbf{i} + v_y\,\mathbf{j}$ or (v_x, v_y). The equations of motion can then be written as a single vector differential equation

Velocity field of a second-order system

$$\frac{d\mathbf{x}}{dt} = \dot{\mathbf{x}} = \mathbf{v}(\mathbf{x}, t). \tag{3.2}$$

3.2.1 Second-order equations and second-order systems

As an example of how the equations of motion of a second-order system may be formed, consider the motion of a particle of unit mass moving along the x-axis subject to a force $F(x, \dot{x}, t)$, depending upon the particle's position x, its velocity $\dot{x}$, and the time. Then, as we reminded you in *Unit 1*, the particle's acceleration is determined by Newton's second law:

$$\frac{d^2x}{dt^2} = \ddot{x} = F(x, \dot{x}, t). \tag{3.3}$$

This is a second-order differential equation; but it can be converted into an equivalent pair of coupled first-order equations by the introduction of a new variable. If we define the new variable y by $y = \dot{x}$ then, since $\dot{y} = \ddot{x}$, Equation 3.3 gives

$$\dot{x} = y, \qquad \dot{y} = F(x, y, t), \tag{3.4}$$

which have the required form. These equations can be written as a vector equation

Since we have taken the mass to be unity, y may be regarded as the linear momentum.

$$\dot{\mathbf{x}} = (y, F(x, y, t)), \tag{3.5}$$

so that the phase velocity is

$$\mathbf{v}(x, y, t) = (y, F(x, y, t)). \tag{3.6}$$

The phase velocity is not just the particle's physical velocity: its first component is the physical velocity $y = \dot{x}$, but its second component is the physical acceleration.

This example illustrates an important point: any second-order differential equation can be written as a pair of coupled first-order equations by the introduction of a new variable in a similar way.

Exercise 3.1

Newton's equation of motion, Equation 3.3, for a particle of unit mass falling freely in a vertical direction in the Earth's gravitational field, assumed to be uniform, is

$$\frac{d^2x}{dt^2} = -g,$$

where x is the vertical height and g is the acceleration due to gravity. What is the phase velocity for the second-order system corresponding to this equation?

Exercise 3.2

Convert the following second-order differential equations into pairs of coupled first-order equations.

(i) $\ddot{x} + \sin x = 0$

(ii) $\ddot{x} + 2a\dot{x} + bx = 0$, where a and b are constants.

(iii) $\ddot{x} + f(x)\dot{x} + g(x) = h(t)$, where f, g and h are arbitrary, sufficiently well-behaved functions.

Exercise 3.3

There is no unique way of converting a second-order equation into a pair of coupled first-order equations. Convert the differential equation

$$\ddot{x} + \dot{x} + x - x^3 = 0$$

into pairs of coupled equations in two ways, first using $y = \dot{x}$, then using $y = \dot{x} + x$.

Exercise 3.4

Consider the second-order differential equation

$$\ddot{x} + x = 0.$$

(i) Show that defining $y = \dot{x}$ reduces it to the pair of equations

$$\dot{x} = y, \qquad \dot{y} = -x.$$

(ii) Verify that the solution of these equations satisfying $x = x_0$, $y = y_0$ when $t = 0$ is

$$x(t) = x_0 \cos t + y_0 \sin t, \qquad y(t) = -x_0 \sin t + y_0 \cos t.$$

(iii) Show that this result may be obtained by introducing polar coordinates in the phase space: show first that if $x = r\cos\theta$, $y = r\sin\theta$, then the equations of motion become

$$\dot{r} = 0, \qquad \dot{\theta} = -1.$$

(iv) Sketch the solution curves.

The device of introducing a new variable may be used to decrease the complexity of a system (at the expense of increasing its order) in other ways: the following exercise gives an example.

Exercise 3.5

Show that by regarding t as a new dependent variable, that is, by setting $y = t$, the non-autonomous first-order system

$$\dot{x} = v(x, t)$$

may be treated as an autonomous second-order system, namely

$$\dot{x} = v(x, y), \qquad \dot{y} = 1.$$

3.2.2 Second-order autonomous systems

Autonomous second-order system

When the velocity field is independent of the time, as is the case for the second-order system in Exercise 3.4 for example, the system is *autonomous*. As for first-order systems, the solution of an autonomous second-order system depends upon the time t and the initial time t_0 only through the difference $t - t_0$. Thus for autonomous systems we can always choose $t_0 = 0$ without any significant loss of generality.

We cannot give a proof of this result by direct construction of the solution as we did for first-order systems. But we can easily adapt one of the other methods of proof used for the corresponding result in the case of a first-order system. This method depends on the property of uniqueness of solutions of systems of simultaneous first-order differential equations. For a second-order system, the *uniqueness theorem* says that for a sufficiently well-behaved velocity field $\mathbf{v}(\mathbf{x}, t)$ there is exactly one vector-valued function $\mathbf{x}(t) = (x(t), y(t))$ which satisfies the system of differential equations

This property is explained in Section 2.1 of *Unit 2*.

Uniqueness theorem for second-order systems

$$\dot{\mathbf{x}} = \mathbf{v}(\mathbf{x}, t),$$

or, equivalently,

$$\dot{x} = v_x(x, y, t), \qquad \dot{y} = v_y(x, y, t),$$

and which also satisfies a given initial condition $\mathbf{x} = \mathbf{x}_0$, or $(x, y) = (x_0, y_0)$, when $t = t_0$. We shall deduce that when the system is autonomous, so that $\mathbf{v}$ does not depend explicitly on t, each solution is a function of $t - t_0$. In fact, we shall show that if $\mathbf{x}(t)$ is the solution which satisfies the initial condition $\mathbf{x} = \mathbf{x}_0$ at $t = 0$, then the solution with initial condition $\mathbf{x} = \mathbf{x}_0$ at $t = t_0$ is $\mathbf{x}(t - t_0)$.

Set $\tilde{\mathbf{x}}(t) = \mathbf{x}(t - t_0)$. Then $\tilde{\mathbf{x}}(t)$ is a solution of the equation of motion, because

$$\frac{d\tilde{\mathbf{x}}}{dt}(t) = \frac{d\mathbf{x}}{dt}(t - t_0) = \mathbf{v}(\mathbf{x}(t - t_0)) = \mathbf{v}(\tilde{\mathbf{x}}(t)).$$

Moreover,

$$\tilde{\mathbf{x}}(t_0) = \mathbf{x}(0) = \mathbf{x}_0,$$

and so $\tilde{\mathbf{x}}(t)$ is the unique solution satisfying the given initial condition. But, by definition, $\tilde{\mathbf{x}}(t) = \mathbf{x}(t - t_0)$ is a vector function of $t - t_0$.

This result extends to systems of arbitrary order; this is the subject of the next exercise. We can use the initial conditions to label the different solutions, as the uniqueness theorem shows; we take advantage of this in setting up the exercise.

Exercise 3.6

Show, more generally, that for an autonomous equation of motion

$$\dot{\mathbf{x}} = \mathbf{v}(\mathbf{x})$$

of any order, if $\mathbf{x}(\mathbf{x}_0, t_0; t)$ is the solution of the equation with initial condition $\mathbf{x} = \mathbf{x}_0$ when $t = t_0$, then $\mathbf{x}(\mathbf{x}_0, t_0; t)$ is a function of $t - t_0$ only. Show further that

$$\mathbf{x}(\mathbf{x}_0, t_1; t) = \mathbf{x}(\mathbf{x}_0, t_0; t - (t_1 - t_0)).$$

During a particular motion of the system the states trace out a continuous curve in the phase space named a *phase curve*. For an autonomous system there is one and only one phase curve through each point of phase space; in the notation of the last exercise, the curve through the point $\mathbf{x}_0$ is given parametrically by $\mathbf{x}(\mathbf{x}_0, 0; t)$. Another solution $\mathbf{x}(\mathbf{x}_0, t_0; t)$ determines the same curve in phase space, but differently parametrised; the difference in the parametrisation is merely a time translation, or change of origin of the time. We do not exclude the possibility that the solution may actually be constant, in the sense that $\mathbf{x}(\mathbf{x}_0, 0; t) = \mathbf{x}_0$ for all t, and it must therefore be borne in mind that the term 'phase curve' includes the degenerate case of a point.

Phase curve

Following the ideas developed for first-order systems in *Unit 2*, given any autonomous system we may define, for each t, a transformation of the phase space ϕ_t by

Section 2.2

$$\phi_t(\mathbf{x}_0) = \mathbf{x}(\mathbf{x}_0, 0; t).$$

Thus $\phi_t(\mathbf{x})$ is found by taking the point $\mathbf{x}$ in phase space and letting it 'flow' along its phase curve for a length of time t. The transformations ϕ_t are collectively known as the *flow* or *phase flow* of the system.

Flow, phase flow

Exercise 3.7

Show that the phase flow ϕ_t of an autonomous system satisfies

$$\phi_t(\phi_s(\mathbf{x})) = \phi_s(\phi_t(\mathbf{x})) = \phi_{s+t}(\mathbf{x})$$

for every pair of real numbers s and t and every point $\mathbf{x}$ of phase space.

When considering a transformation of phase space it is more natural to denote the general point on which it acts by $\mathbf{x}$ than by $\mathbf{x}_0$.

Exercise 3.8

Verify by an explicit calculation that the result of Exercise 3.7 holds for the flow of the system discussed in Exercise 3.4 (page 45).

Exercise 3.9

The equations of motion for the particle falling freely under gravity (Exercise 3.1, page 44) are

$$\dot{x} = y, \qquad \dot{y} = -g.$$

Show that if (x_0, y_0) is the state at time $t = 0$ then

$$x(t) = x_0 + y_0 t - \tfrac{1}{2}gt^2, \qquad y(t) = y_0 - gt.$$

Write down the flow of this system, and confirm by an explicit calculation that it satisfies the property given in Exercise 3.7.

Thus, just as in the first-order case, the flow may be thought of as representing the effect of time-translations on phase space.

3.3 Phase diagrams of second-order autonomous systems

Second-order autonomous systems display more complicated behaviour than the first-order systems dealt with in *Unit 2*. However, as the phase space can be regarded as a two-dimensional surface, which in many cases is a plane, they are easier to understand than higher-order systems; in particular, we can still hope to use diagrams and graphs to represent the motion. In this section and the next we extend the ideas we described in *Unit 2* to develop graphical methods for the analysis of second-order systems.

Section 2.3

3.3.1 Phase portraits and phase diagrams

A complete understanding of the motion of a second-order autonomous system could be obtained by drawing all of its possible phase curves. The collection of all the phase curves is known as the *phase portrait*, or *phase diagram*, of the system.

Phase portrait

Phase diagram

Of course, in practice it is impossible to draw *all* the phase curves; but generally only a representative sample is required. However, even for second-order systems the equations of motion are rarely soluble in closed form, so qualitative methods of investigating phase portraits are important. For two-dimensional systems there is a simple geometric procedure for obtaining a picture of the phase portrait without solving the equations of motion; it is closely related to the method of direction fields, as we shall explain later.

The procedure is the following. Take a suitable grid of points in the phase plane, and at each point $\mathbf{x}_k$ draw an arrow in the direction of the phase velocity at that point whose length is proportional to $|\mathbf{v}(\mathbf{x}_k)|$; we use the convention that the base of the arrow is at $\mathbf{x}_k$. In other words, at each grid point $\mathbf{x}_k$ one draws the phase velocity vector $\mathbf{v}(\mathbf{x}_k)$. The resulting set of arrows can often be joined 'by eye' to construct a set of approximate phase curves.

Example 3.1

Draw a phase diagram for the system

$$\dot{\mathbf{x}} = \mathbf{v}(\mathbf{x}), \qquad \text{where} \quad \mathbf{v}(x, y) = (y, -g).$$

(This system corresponds to the equation of motion of a particle of unit mass falling freely in a vertical direction in the Earth's gravitational field, assumed to be uniform, as was shown in Exercise 3.1, page 44.)

Solution

It is relatively easy to construct a phase diagram in this case, because $\mathbf{v}$ is independent of x, so that for a given y all arrows are parallel and of the same length. Further, as the y-component of $\mathbf{v}$ is negative the arrows always point downwards, and when $y = 0$ they are all parallel to the y-axis; for $y > 0$ they are in the direction of increasing x, and for $y < 0$ decreasing x. Thus the phase diagram is as shown in Figure 3.1.

An individual phase curve can be drawn by following the arrows and interpolating where necessary. Note that wherever the phase curve passes through a grid point the arrow based at the point should be tangent to the curve. ☐

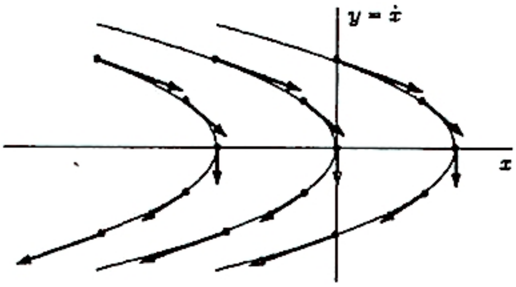


Figure 3.1 Phase diagram for the system with $\mathbf{v}(x, y) = (y, -g)$.

The solution of the equations of motion for the particle falling freely under gravity is given in Exercise 3.9; it is

$$x(t) = x_0 + y_0 t - \tfrac{1}{2} g t^2, \qquad y(t) = y_0 - gt, \tag{3.7}$$

where (x_0, y_0) is the state at time $t = 0$. These equations give the state of the system at any time t, and by drawing the curve $(x(t), y(t))$, parametrised by the time t, we can construct the phase curve though the point (x_0, y_0), and so check the sketch in Figure 3.1. When the time is eliminated from Equations 3.7 we obtain

$$y^2 + 2gx = y_0^2 + 2gx_0, \tag{3.8}$$

which is the equation of a parabola with its axis parallel to the x-axis. Thus the phase curves are parabolas, which confirms Figure 3.1.

Parabolas seem to turn up all over the place in the consideration of motion under uniform gravity, and so it may be as well to remind you exactly which system we are discussing here. The system describes the motion of a particle *along a vertical straight line* under the influence of gravity; it is a second-order system because the equation of motion is a second-order differential equation. The axes in Figure 3.1 correspond to the particle's position (x-axis) and velocity (y-axis). Do not confuse this diagram with the one for the motion of a particle under gravity in two dimensions.

Exercise 3.10

How can the particle's actual motion be inferred from the phase diagram in Figure 3.1?

Exercise 3.11

Describe the phase diagrams for the systems with the following velocity fields.

(i) $\mathbf{v}(\mathbf{x}) = (a, b)$, where a and b are positive constants.

(ii) $\mathbf{v}(\mathbf{x}) = (a, x)$, where $a \neq 0$.

Find equations for the phase curves.

The next example is rather harder.

Example 3.2

Sketch the phase diagram for the system with the velocity field

$$\mathbf{v}(\mathbf{x}) = (x^2, y).$$

Find equations for its phase curves.

Solution

The following points are useful in obtaining an idea of the phase diagram. The x-component of the phase velocity is always positive, and it increases as $|x|$ increases. The y-component of the phase velocity changes sign with y; in fact, the phase diagram for $y < 0$ may be obtained from that for $y > 0$ by reflection in the x-axis. The system is separated, in the sense that the x-component of the phase velocity is a function of x alone, and its y-component is a function of y alone. This means that all points with the same x-coordinate have the same x-component of phase velocity, and similarly all points with the same y-coordinate have the same y-component of phase velocity. On the x-axis the phase velocity points along the x-axis in the positive x-direction, while on the y-axis the phase velocity points up the y-axis for $y > 0$, down it for $y < 0$. At the origin the phase velocity is zero.

Some representative phase velocity arrows are shown below.

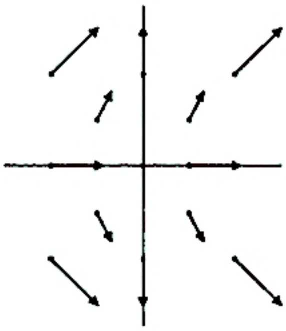


Figure 3.2 Representative phase velocities for the system with $\mathbf{v}(x, y) = (x^2, y)$.

Since the system is separated it is relatively easy to find the phase curves by solving the equations of motion $\dot{x} = x^2$, $\dot{y} = y$. In fact, the solutions of each of these equations were obtained in *Unit 2*; they are

Exercises 2.21, 2.26 and 2.3

$$x(t) = \frac{x_0}{1 - x_0 t}, \qquad y(t) = y_0 e^t.$$

From these we see that the phase curves are always directed away from the x-axis, and that those for $y < 0$ can be obtained by reflection in the x-axis from those for $y > 0$, as we noted above in relation to the phase diagram. We therefore consider only the phase curves in the upper half-plane.

A phase point initially in the first quadrant, $x_0 > 0$, $y_0 > 0$, has $\dot{x} > 0$ and $\dot{y} > 0$; the x-motion terminates at $t = 1/x_0$, when $x(t) \to \infty$, while $y(t) \to y_0 \exp(1/x_0)$. As $t \to -\infty$, $x(t)$ behaves like $1/|t|$, and so $x(t) \to 0$; $y(t)$, on the other hand, behaves like e^t, and so $y(t) \to 0$ also, but much faster than $x(t)$, which results in the phase curves shown in the first quadrant of Figure 3.3.

For a starting point in the second quadrant, $x_0 < 0$, $y_0 > 0$, the x-motion terminates at $t = -1/|x_0|$ with $x(t) \to -\infty$ and $y(t) \to y_0 \exp(-1/|x_0|)$. As $t \to \infty$, so $y(t) \to \infty$, but $x \to 0$, giving the phase curves shown in the second quadrant of Figure 3.3.

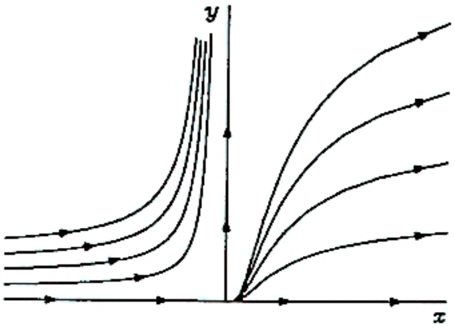


Figure 3.3 Phase curves for the system $\dot{x} = x^2$, $\dot{y} = y$, with $y \geq 0$.

Note that if $y_0 = 0$ then the solution reduces to the one-dimensional system $\dot{x} = x^2$ on the x-axis, whose phase diagram we know from *Unit 2* (see Figure 2.6). Similarly, if $x_0 = 0$ the system becomes $\dot{y} = y$ on the y-axis, whose solution was discussed extensively in *Unit 2*. Finally, if $x_0 = y_0 = 0$ then $\mathbf{x}(t) = 0$ for all time; the origin is a fixed point for the motion along both the x-axis and the y-axis, and is indeed a fixed point for the whole two-dimensional motion. ☐

Subsection 2.3.2; see Example 2.7 in particular.

Exercise 3.12

Sketch the phase diagram for the system with the velocity field

$$\mathbf{v}(\mathbf{x}) = (x, -y).$$

Find equations for its phase curves.

In all of the preceding examples and exercises we have been able to solve the equations of motion for the system concerned, and therefore draw the phase curves. We emphasise that this was done for the purpose of illustration only; in practice the phase diagram is used in order to avoid the necessity of solving the equations of motion, if indeed it is possible to solve them in closed form.

The various concepts introduced in this section and at the end of the previous subsection may be linked together by means of an analogy between an autonomous dynamical system and the steady flow of a fluid, in an extension of an idea which we introduced for first-order systems in *Unit 2*. We must now consider the flow of a fluid in two dimensions: a thin layer of fluid spread over a horizontal plane surface, say. We assume that the particles of the fluid move only horizontally.

Subsection 2.3.1

First, imagine concentrating your attention on a fixed point of the plane. As time passes, a succession of particles of the fluid passes over this point. The phase velocity at the point gives the instantaneous velocity of the particle passing over the point at any instant of time. If the system is autonomous, then as the particles pass over the point, they always have the same instantaneous velocity.

Next, allow your eye to follow a particle after it passes the point you have been observing. The path it follows is a phase curve of the point. You could imagine that the particle is visually tagged in some way: perhaps it is a speck of fluorescent paint carried along by the fluid. Then time-elapse photography (in which the shutter of a still camera is left open for a period of time) will reveal its path, as a curve in phase space; but a complete specification of the motion requires a method of recording where the particle is located on its phase curve at any instant of time – a movie camera, in other words. If the system is autonomous then the same picture will be obtained whenever it is taken.

If a number of points are tagged in the same way then time-elapse photography will give a picture of the phase portrait; this is the kind of picture one sometimes sees of wind-tunnel experiments. Snapshots separated by a time t, on the other hand, represent the flow ϕ_t of the system.

3.3.2 Phase diagrams, phase curves and direction fields

In Equation 3.8 (page 48) we gave the equations of the phase curves of the system considered in Example 3.1 (motion under uniform gravity, page 48) in terms of x and y alone, by eliminating the parameter t from the solutions of the equations of motion. It is sometimes possible to obtain the equations of the phase curves in this form directly, by converting the two coupled differential equations for the motion (Equations 3.1, page 44) into a single first-order equation for the curve. This is achieved by simply dividing one equation of motion by the other and using the result

$$\frac{\dot{y}}{\dot{x}} = \frac{dy/dt}{dx/dt} = \frac{dy}{dt}\frac{dt}{dx} = \frac{dy}{dx} \tag{3.9}$$

to obtain

$$\frac{dy}{dx} = \frac{v_y(x,y)}{v_x(x,y)}. \tag{3.10}$$

Notice that normally this equation has meaning only if the system is autonomous.

By writing the equation in this way we have implicitly assumed y to be a function of x. Obviously, we could choose y to be the independent variable and write the equation in the equivalent form

$$\frac{dx}{dy} = \frac{v_x(x,y)}{v_y(x,y)}. \tag{3.11}$$

In particular, in any neighbourhood of a point where v_x vanishes, Equation 3.11 must be used rather than Equation 3.10 – unless, of course, v_y vanishes simultaneously with v_x. Points which are zeros of the velocity field – points at which both v_x and v_y vanish – must be treated separately, and are discussed in Section 3.4.

For example, in the case of motion under uniform gravity, Equation 3.10 for the phase curve is

$$\frac{dy}{dx} = -\frac{g}{y} \qquad \text{or} \qquad \int_{y_0}^{y} dy\, y = -g\int_{x_0}^{x} dx,$$

whence

$$\tfrac{1}{2}(y^2 - y_0^2) = -g(x - x_0),$$

which is equivalent to Equation 3.8 (page 48).

Equation 3.8 specifies the phase curves in sufficient detail to confirm the sketch in Figure 3.1 (page 48). Equations 3.7 (page 48) give the same set of curves but in parametric form: both x and y are given as functions of a parameter, in this case the time. You will notice immediately that these equations contain more information; they describe where the phase point is at a given time, whereas the explicit equation for the phase curve, Equation 3.8, prescribes only the path, not the history; even the direction of motion is lost in this equation. A similar loss of information will always occur when the original equations of motion are replaced by Equation 3.10 or Equation 3.11.

Exercise 3.13

Derive the equation for the phase curves for the system with velocity field

$$\mathbf{v}(\mathbf{x}) = (x^2, y),$$

which was treated in Example 3.2 (page 49), in the form

$$y = y_0 e^{1/x_0} e^{-1/x}, \qquad x_0 \neq 0.$$

Exercise 3.14

Find the phase curves for the system

$$\dot{x} = 2xy, \qquad \dot{y} = y^2 - x^2.$$

[Hint: Equation 3.10 for this case may be simplified by the substitution $y = xz$, which leads to a differential equation relating z and x which is separable.]

Let us now consider phase diagrams for autonomous second-order systems in relation to direction fields for first-order differential equations. The relation between the two is completely analogous to the relation between the system of differential equations

$$\dot{x} = v_x(x,y), \qquad \dot{y} = v_y(x,y) \tag{3.12}$$

and the single differential equation (Equation 3.10)

$$\frac{dy}{dx} = \frac{v_y(x,y)}{v_x(x,y)}. \tag{3.13}$$

To construct the phase diagram for Equations 3.12 we draw, at the point (x, y) in the phase plane, the vector $v_x(x,y)\,\mathbf{i} + v_y(x,y)\,\mathbf{j}$. The direction of this vector is determined by the ratio $v_y(x,y)/v_x(x,y)$, which specifies the value of the direction field at (x, y) for Equation 3.13. Thus the direction field is simply the field of directions defined by the vector field $\mathbf{v}(x,y)$, and in going from the phase diagram to the diagram of the direction field one ignores the lengths of the vectors but preserves their directions. This is completely in accord with the fact that Equation 3.13 determines the solution curves of the system as geometrical entities in the phase plane, but without specifying how any solution curve is traversed. If we know how a point moves along a curve, in other words if the curve is given parametrically in terms of time, then we know the instantaneous speed of the motion as well as the instantaneous direction. This is the extra information in Equations 3.12 compared with Equation 3.13, and also in the phase diagram compared with the diagram of the direction field.

3.4 Fixed points, equilibrium and stability

A point $\mathbf{x}_f$ where the phase velocity $\mathbf{v}(\mathbf{x}_f)$ is zero is named a *fixed point*, as a system initially in the state $\mathbf{x}_f$ stays there. The fixed points are thus the roots of the equation $\mathbf{v}(\mathbf{x}) = 0$:

Fixed point

$$\mathbf{v}(\mathbf{x}_f) = 0 \iff \mathbf{x}_f \text{ is a fixed point.} \tag{3.14}$$

The fixed points represent states in which the system is in equilibrium; for this reason some texts name them *equilibrium points*. A system in equilibrium stays in that state; but there is a negligible chance of it being precisely in equilibrium in the first place, so its behaviour is determined by the stability of the fixed point.

Equilibrium point

All this is superficially similar to the behaviour of first-order systems; however, the stability analysis of fixed points for second-order systems is more subtle than that for first-order systems. As an example, consider a vertical, circular, smooth wire ring on which slides a small bead, the only significant force being that of gravity. There are clearly two fixed points, one at the top and one at the bottom of the ring. If the bead is disturbed from the fixed point at the top of the ring then it initially moves away, and so the behaviour of the system near this fixed point may appear to be similar to that of a first-order system near an unstable fixed point. However, the bead will slide around the ring and eventually approach the fixed point again from the other side. On the other hand, a small displacement of the system from the lower fixed point results in perpetual oscillatory motion about it (assuming there is no friction). So there is no analogue amongst first-order systems for the behaviour of this system near either of its fixed points.

This is an example of motion under Newton's laws in one dimension, and it is therefore a second-order system. A pendulum, consisting of a particle attached to a rigid rod free to rotate in a vertical plane, would show the same behaviour.

Just as in the first-order case, it is desirable to be able to understand the motion in the vicinity of fixed points. We shall tackle this problem in the next section; but first we shall explain some concepts which will help us define more precisely the various types of stability.

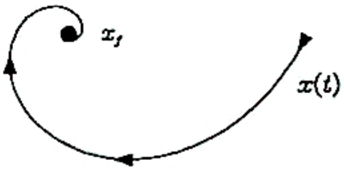
A fixed point $\mathbf{x}_f$ is said to be an *attractor* of some motion $\mathbf{x}(t)$ if

Attractor of a motion

$$\lim_{t \to +\infty} \mathbf{x}(t) = \mathbf{x}_f. \tag{3.15}$$

Thus a phase curve which has an attractor might look something like Figure 3.4.

More generally, attractors can be sets of points; we shall discuss this possibility later (Section 3.7).



Attractor

Figure 3.4 A phase curve with an attractor.

Note the importance of the direction of time in the definition of an attractor. A fixed point which is an 'anti-attractor' for some motion, that is, a fixed point $\mathbf{x}_f$ such that

$$\lim_{t \to -\infty} \mathbf{x}(t) = \mathbf{x}_f,$$

is called a *repellor* of the motion $\mathbf{x}(t)$.

Repellor

A fixed point may be an attractor for some motions passing close to it but not for others. This type of behaviour occurs also for first-order systems, in the case of the velocity function $v(x) = cx^2$ for example; there is clearly room for more variations on this theme in a two-dimensional phase space.

See *Unit 2* Subsection 2.3.2.

Exercise 3.15

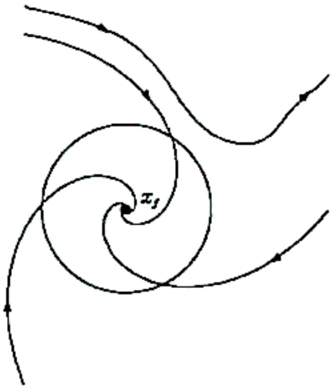
Is it possible for a single fixed point to be simultaneously an attractor and a repellor of the same motion?

Suppose, however, that $\mathbf{x}_f$ is a fixed point which is an attractor of *all* motions which pass sufficiently close to it; then if the system is initially in the state $\mathbf{x}_f$ and is slightly disturbed it will tend to revert to that state, and so the fixed point $\mathbf{x}_f$ certainly deserves to be called stable. In fact, for reasons which will become clear shortly, such a fixed point is called strongly stable. Thus a fixed point $\mathbf{x}_f$ is *strongly* or *asymptotically stable* if it is an attractor for *all* phase curves passing through some neighbourhood of it. Figure 3.5 illustrates this concept.

Strongly or asymptotically stable fixed point

A fixed point which is a repellor of all motions which pass sufficiently close to it is said to be *strongly* or *asymptotically unstable*. A picture of a strongly unstable fixed point may be obtained by reversing the directions of the arrows on the phase curves in Figure 3.5.

Strongly or asymptotically unstable fixed point



Strongly stable fixed point

Figure 3.5 A strongly stable fixed point.

Exercise 3.16

MST204 Unit 8

The equations of motion for a damped harmonic oscillator may be written

$$\dot{x} = y, \qquad \dot{y} = -\omega^2(x - x_f) - 2\alpha\omega y,$$

where α, ω and x_f are constants, with the damping factor α and the natural frequency ω positive. The system has a fixed point at $(x, y) = (x_f, 0)$. By using what you know about the qualitative features of the motion of such a system, show that this fixed point is strongly stable.

Now the fixed point at the bottom of the ring in the example at the beginning of this section is not an attractor for any motion (other than the equilibrium point itself). But if the bead is slightly disturbed from that equilibrium position it subsequently certainly stays close to it, so it is also proper to call this fixed point stable. Thus not all stability is strong stability, and for the definition of stability in general we need to take account of the kind of behaviour exemplified by the bead and ring. We therefore make the following subsidiary definition: a region R_1 of phase space is said to *contain* the motion from some other region of phase space R_2, if *every* motion originating in R_2 at time $t = 0$ remains within R_1 for all $t \geq 0$, that is, if

Contained motion

$$\mathbf{x}(0) \in R_2 \Longrightarrow \mathbf{x}(t) \in R_1 \qquad \text{for all } t \geq 0. \tag{3.16}$$

The definition requires that R_2 is itself contained in R_1, but permits R_1 to be bigger than R_2. It also allows for the possibility that there are motions passing through the larger set R_1 which do not remain within it. The basic idea is that motions starting in R_2 cannot move too far away from it. The following figure illustrates these possibilities.

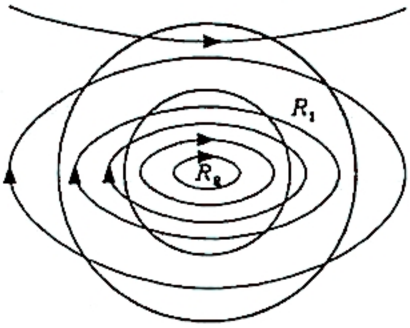


Figure 3.6 The region R_1 contains the motion from R_2. Here R_2 is the region inside the smaller circle, while R_1 is the interior of the larger circle.

Consider those motions in Figure 3.6 which at some time or other lie within R_2. Some of them remain entirely within R_2, some do not; but all motions which are ever inside R_2 are *always* inside R_1. Thus R_1 contains the motion from R_2.

Exercise 3.17

Formulate the definition of one region containing the motion from another in terms of the flow ϕ_t of the system.

A fixed point $\mathbf{x}_f$ is *stable* if every neighbourhood of $\mathbf{x}_f$ contains the motion from some other neighbourhood of $\mathbf{x}_f$. Figure 3.6 shows the flow around a fixed point that is stable but not strongly stable. Such stability is important for the Hamiltonian systems of the next unit. All strongly stable points are stable, but not vice versa.

Stable fixed point

A fixed point that is not stable is *unstable*. All strongly unstable fixed points are unstable, but not vice versa.

Unstable fixed point

Exercise 3.18

For the system whose phase diagram is shown in Figure 3.7, with the indicated fixed points, identify

(i) the fixed points which attract *some* motion;

(ii) the fixed points which repel *some* motion;

(iii) the stable fixed points;

(iv) the strongly stable fixed points;

(v) the unstable fixed points, distinguishing any which are strongly unstable.

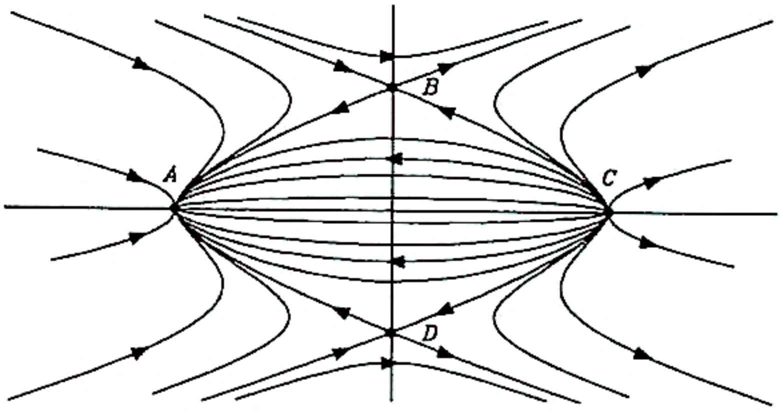


Figure 3.7

Exercise 3.19

Use the result of Exercise 3.4 (page 45) to show that the origin of the phase space is a stable, but not a strongly stable, fixed point of the second-order system determined by the differential equation

$$\ddot{x} + x = 0.$$

Exercise 3.20

Consider the system with equations of motion

$$\dot{x} = y, \qquad \dot{y} = x,$$

corresponding to the second-order differential equation $\ddot{x} - x = 0$.

(i) Show that the origin is a fixed point of this system.

(ii) Show that the solution of the system which satisfies the initial conditions $(x(0), y(0)) = (x_0, y_0)$ is

$$x(t) = x_0 \cosh t + y_0 \sinh t, \qquad y(t) = x_0 \sinh t + y_0 \cosh t.$$

(iii) Deduce that the origin is an attractor for the motion of any point on the line $y = -x$, and a repellor for points on the line $y = x$. Conclude that the origin is an unstable fixed point.

The terminology introduced in this section is applicable to first-order systems as well as to second-order ones; but we did not use it in *Unit 2* because the ideas of stability are more straightforward for first-order than for second-order systems, so some of the distinctions necessary here are not required in the simpler case. In particular, there is no direct analogue of behaviour of the type illustrated in Figure 3.6 amongst first-order systems, so we did not make the distinction between strong stability and other kinds of stability in *Unit 2*. Nevertheless, it is useful to know how terms such as 'attractor' apply to first-order systems.

Exercise 3.21

For the first-order systems

(i) $\dot{x} = x(a-x)(b-x), \quad x \geq 0,\ a > b > 0,$

(ii) $\dot{x} = \sin(x^2),$

find the fixed points, and determine which are attractors and which are repellors; in the former case determine the neighbourhoods which are attracted to the fixed point.

A particular class of systems for which the determination of the stability of fixed points is relatively straightforward consists of those whose equations of motion are *separated*. By this we mean that the equations of motion have the form

Separated system

$$\dot{x} = v_x(x), \qquad \dot{y} = v_y(y), \tag{3.17}$$

where v_x is a function of x only and v_y is a function of y only.

The systems considered in Example 3.2, page 49, and Exercise 3.12, page 50, are of this type.

The immediate benefit of having the equations of motion separated is that we can integrate them. But in addition the fixed point theory for first-order systems can be used to help us investigate the nature of a fixed point of the second-order system, because we can separate the motion into that of two uncoupled first-order systems.

It clearly follows from the equations of motion of the separated system, Equations 3.17, that if $x = x_f$ is a fixed point of one first-order system and $y = y_f$ is a fixed point of the other, then

$$\mathbf{x}_f = (x_f, y_f) \tag{3.18}$$

is a fixed point of the combined second-order system. Furthermore, the flows of the first-order systems in the neighbourhoods of $x = x_f$ and $y = y_f$ determine the flow of the second-order system in the neighbourhood of $\mathbf{x}_f$. Thus we can use the theory of fixed points of first-order systems to help understand the fixed points of the second-order system.

Unit 2 Subsection 2.3.2

Exercise 3.22

Consider the second-order system with equations of motion given by Equations 3.17 and with fixed point (x_f, y_f). Suppose that $x = x_f$ and $y = y_f$ are simple fixed points of the two first-order systems. Show that the solution of Equations 3.17 with initial condition $(x, y) = (x_0, y_0)$ at $t = 0$, where (x_0, y_0) is near (x_f, y_f), is given approximately by

$$x(t) = x_f + (x_0 - x_f)e^{bt}, \qquad y(t) = y_f + (y_0 - y_f)e^{ct},$$

where $b = v_x'(x_f)$ and $c = v_y'(y_f)$.

Deduce that if x_f and y_f are both stable then (x_f, y_f) is strongly stable, and state the corresponding results for the other possibilities for x_f and y_f.

It is often the case that equations of motion which are not separated initially may be made so by a change of coordinates. We say that an autonomous second-order system is *separable* if there is a transformation to new coordinates in terms of which the equations of motion become separated. The only means of knowing whether a coordinate system exists in which a given system is separable is to actually construct it; there are no general rules to help with this, though usually understanding the symmetries of the system helps. The system considered in Exercise 3.20 is an example of a separable system, as is demonstrated in the solution. Later we shall consider systems which are separable in oblique coordinates and systems which are separable in polar coordinates.

Separable system

3.5 Classification of fixed points of linear systems

We turn now to the classification of fixed points. The idea is to classify a fixed point according to the qualitative nature of the flow in its vicinity; this goes further than just deciding whether the fixed point is stable or unstable. However, the classification would become extremely complicated if we were to consider every conceivable case, so from the outset we shall limit the type of fixed point under consideration. In particular, we do not want to have to consider situations in which there may be a whole line of fixed points, for example. The following definition is designed to eliminate this possibility.

A fixed point is said to be *isolated* if there is some neighbourhood of it which contains no other fixed points.

Isolated fixed point

There are, in fact, ten types of isolated fixed point for autonomous second-order systems, so the general theory is a little more complicated than it is for first-order systems. But each possibility occurs for a suitable linear system, just as in the first-order case. We shall begin by describing the types of fixed point of linear systems, since they are the simplest; but our results can be used to classify the fixed points of non-linear systems, as we shall show in the next section.

3.5.1 Linear systems

By a *linear system* we mean here a system whose equations of motion are linear constant-coefficient homogeneous equations, which may therefore be written, for a second-order system, in the form

Linear system

$$\dot{x} = ax + by, \qquad \dot{y} = cx + dy, \tag{3.19}$$

where a, b, c and d are constants; or in the matrix form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}, \tag{3.20}$$

where $\mathbf{A}$ is the real 2×2 matrix

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}. \tag{3.21}$$

Such a system has a fixed point at the origin.

The flows of second-order linear systems are discussed in *M203 Block B Unit 4*. If you have studied that course you should find much of the material in this section familiar.

Exercise 3.23

Show that the fixed points of the system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ consist of those vectors $\mathbf{x}_f$ such that

$$\mathbf{A}\mathbf{x}_f = 0.$$

Conclude that if the matrix $\mathbf{A}$ is non-singular then the fixed point at the origin is isolated; but that if $\mathbf{A}$ is singular the system will have, at best, a whole line of fixed points passing through the origin, and therefore no isolated fixed points at all.

Since we are studying the behaviour of systems near isolated fixed points we shall assume that $\mathbf{A}$ is non-singular and that the origin is the only fixed point. A linear system with non-singular matrix $\mathbf{A}$ is named *simple*: a simple linear system has a single, isolated, fixed point at the origin.

Simple linear system

The solution of linear systems was described in *Unit 1* Section 1.4. The method adopted there was to find a non-singular matrix $\mathbf{P}$ which would transform $\mathbf{A}$ into one of the three possible standard forms of 2×2 matrices, and to use this transformation to simplify the equations. This has the same effect as carrying out a coordinate transformation to oblique coordinates. Now the essential nature of the fixed point – whether it is stable or unstable, for example – is quite independent of whether the coordinates in terms of which the equations of motion are written are Cartesian or not. Thus in order to classify the types of isolated fixed point of linear systems it is sufficient to examine the systems for which the coefficient matrix is in standard form.

The standard form appropriate to **A** depends on the number and nature of the eigenvalues and eigenvectors of **A**; thus the eigenvalues and eigenvectors of **A** will play the central role in determining the stability of the fixed point of the linear system. In this context it is worth remembering that a matrix with an eigenvalue which is zero is singular; since we are concerned only with isolated fixed points, we may assume that the eigenvalues of **A** are non-zero.

The following exercise provides further evidence of the importance of the eigenvalues and eigenvectors of **A** in determining the phase portrait of the system.

Exercise 3.24

Show that if $\mathbf{u}$ is an eigenvector of $\mathbf{A}$ with eigenvalue λ then the solution of $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ with initial condition $\mathbf{x} = \mathbf{u}$ when $t = 0$ is

$$\mathbf{x}(t) = e^{\lambda t}\mathbf{u}.$$

Deduce that the eigenlines of **A** (the lines through the origin in the directions of the eigenvectors of **A**) are phase curves of the system, and that the origin is an attractor or a repellor of the corresponding phase curve, depending on whether λ is negative or positive.

As a first step we shall obtain the phase portraits of all linear systems in which the coefficient matrix is already in standard form. The phase portraits of the more general linear systems may then be derived from these by transforming to oblique coordinates. We shall deal with the three standard forms one by one.

3.5.2 Two real distinct eigenvalues

Suppose that

$$\mathbf{A} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \tag{3.22}$$

where λ_1 and λ_2 are real and distinct, and neither is zero. The equations of motion are simply

$$\dot{x} = \lambda_1 x, \qquad \dot{y} = \lambda_2 y. \tag{3.23}$$

The equations are separated; their solutions are

$$x(t) = x_0 e^{\lambda_1 t}, \qquad y(t) = y_0 e^{\lambda_2 t}. \tag{3.24}$$

The phase curves of the second-order system are obtained by eliminating t from these equations, to obtain (for $x_0 \neq 0$)

$$y = y_0 \left(\frac{x}{x_0}\right)^{\lambda_2/\lambda_1}. \tag{3.25}$$

Note that, from Equations 3.24, x always has the same sign as x_0, and y the same sign as y_0. Thus each phase curve with $(x_0, y_0) \neq (0, 0)$ lies entirely within one of the four quadrants of the phase plane. Furthermore, the positive and negative x- and y-axes are also phase curves (these are the eigenlines); also, the origin is a degenerate phase curve because it is a fixed point.

Compare Exercise 3.24.

The shapes of the phase curves for non-zero x_0 and y_0 depend in part upon the stability of equilibrium in each sub-system; there are three possibilities according as both sub-systems are stable, both are unstable, or one is stable and the other is unstable.

Compare Exercise 3.22, page 56.

If both sub-systems are stable, $\lambda_1 < 0$ and $\lambda_2 < 0$, then $\lambda_2/\lambda_1 > 0$. If, further, $|\lambda_2| > |\lambda_1|$, then the curve given by Equation 3.25 is roughly the shape of a parabola, with its axis along the y-axis; indeed, $y = kx^2$ is a particular case, corresponding to $\lambda_2 = 2\lambda_1$. If $|\lambda_2| < |\lambda_1|$, on the other hand, then the curve has a shape like that of the square-root curve $y = k\sqrt{x}$; but this is not essentially different from the first case, since it amounts simply to interchanging the roles of x and y.

As $t \to +\infty$, $\mathbf{x}(t) \to 0$ along every phase curve; thus the fixed point is *strongly stable.* The phase portrait for $\lambda_2 < \lambda_1 < 0$ looks like Figure 3.8.

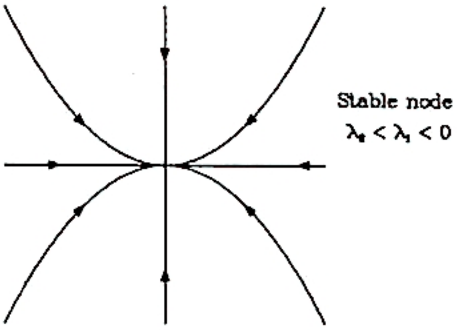


Figure 3.8 Stable node, $\lambda_2 < \lambda_1 < 0$.

Note that this figure shows *eight* phase curves (not counting the degenerate one at the origin). If $\lambda_1 < \lambda_2 < 0$, the phase portrait is obtained by rotating this one through a right-angle. In either case the fixed point is called a *stable node.*

Stable node

If both sub-systems are unstable, $\lambda_1 > 0$ and $\lambda_2 > 0$, the phase curves are similar to those in the previous case, but they are described in the opposite sense. Thus the fixed point is *strongly unstable.* The phase portrait for $\lambda_2 > \lambda_1 > 0$ looks like Figure 3.9.

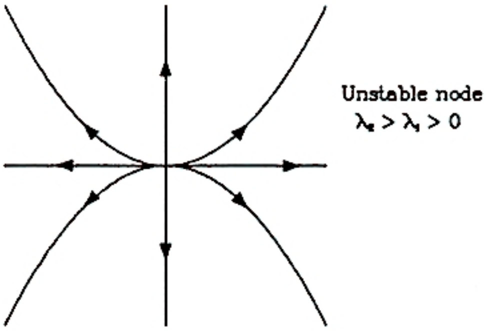


Figure 3.9 Unstable node, $\lambda_2 > \lambda_1 > 0$.

If $\lambda_1 > \lambda_2 > 0$ we obtain curves of the same general shape but rotated through a right-angle. This type of fixed point is called an *unstable node.*

Unstable node

If one of the sub-systems is stable and the other unstable, for instance if $\lambda_1 < 0 < \lambda_2$, then the phase curves are similar to hyperbolas $y = k/x$; indeed, they are exactly hyperbolas if $\lambda_1 = -\lambda_2$. With the given choice of signs of λ_1 and λ_2, motion on the x-axis is attracted towards the origin, and motion on the y-axis is repelled away from the origin. The phase portrait looks like Figure 3.10.

This case is considered in Exercise 3.12, page 50.

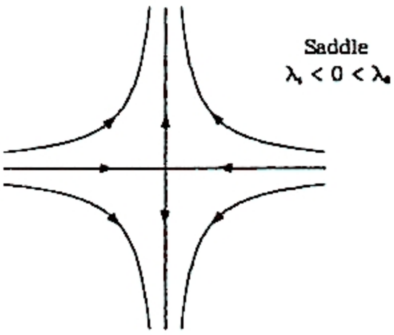


Figure 3.10 Saddle, $\lambda_1 < 0 < \lambda_2$.

For $\lambda_2 < 0 < \lambda_1$ the fixed point attracts the y-axis and repels the x-axis, so the phase portrait differs by having the directions of all the arrows reversed. In either case the fixed point is called a *saddle*, or sometimes a *hyperbolic point*, and is *unstable*.

Saddle

Hyperbolic point

3.5.3 Real identical eigenvalues

When the eigenvalues of the coefficient matrix A are real and equal the matrix can have one of two standard forms, depending on how many independent eigenvectors there are. There are thus two separate cases to consider. As before, we assume that A is non-singular, so the eigenvalue is non-zero.

First, if there are two linearly independent eigenvectors then

$$\mathbf{A} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}. \tag{3.26}$$

The equations of motion are those of Equations 3.23 (page 58), but with $\lambda_1 = \lambda_2 = \lambda$. The solutions are

$$x(t) = x_0 e^{\lambda t}, \qquad y(t) = y_0 e^{\lambda t}. \tag{3.27}$$

The phase curves are just the straight lines

$$\frac{y}{y_0} = \frac{x}{x_0}. \tag{3.28}$$

If $\lambda < 0$ the fixed point is *strongly stable*, and is called a *stable star*; its phase portrait looks like Figure 3.11.

Stable star

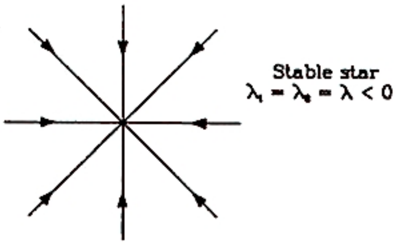


Figure 3.11 Stable star, $\lambda_1 = \lambda_2 = \lambda < 0$.

If $\lambda > 0$ the fixed point is *unstable*; it is called an *unstable star*, and its phase portrait looks like that of a stable star but has the arrows reversed.

Unstable star

In the second case there is one eigenvector, up to scalar multiples, corresponding to the single eigenvalue λ. Then

$$\mathbf{A} = \begin{bmatrix} \lambda & 0 \\ 1 & \lambda \end{bmatrix}. \tag{3.29}$$

The equations of motion are then

$$\dot{x} = \lambda x, \qquad \dot{y} = x + \lambda y, \tag{3.30}$$

and the solution is

Unit 1 Section 1.5, Exercise 1.59

$$x(t) = x_0 e^{\lambda t}, \qquad y(t) = (y_0 + x_0 t)e^{\lambda t}. \tag{3.31}$$

Since $x(t)$ always has the same sign as x_0 for $x_0 \neq 0$, the phase curve has the equation

$$y = x\left(\frac{y_0}{x_0} + \frac{1}{\lambda}\ln\left(\frac{x}{x_0}\right)\right). \tag{3.32}$$

The two halves of the y-axis are also phase curves, corresponding to the eigenvector $[0 \quad 1]^T$.

As an example, we analyse the phase curve for $x_0 = y_0 = 1$, $\lambda = -1$, whose equation is

$$y = x(1 - \ln x), \qquad \text{and} \qquad \frac{dy}{dx} = -\ln x.$$

In fact, this one case is enough to give the general shape of any phase curve; the other curves may be obtained from this one by a combination of stretching, reflecting and rotating it. Note that the phase curve remains in the right half-plane, and so the argument of the logarithm is always positive. As $x \to 0$, $\ln x \to -\infty$, but $x \ln x \to 0$, and so the phase curve approaches the origin from the right with ever-increasing gradient as $x \to 0$, which means that the curve approaches the origin tangentially to the y-axis. As $x \to +\infty$, $y \to -\infty$ and $dy/dx \to -\infty$, which means that the curve becomes more nearly parallel to the y-axis as x increases. But $y = 0$ where $x = e$; also $y > 0$ for $x < e$ and $y < 0$ for $x > e$, so there is a turning point, which is clearly a maximum, where $x = 1$, at which point $y = 1$. For negative λ all phase curves with $x_0 > 0$ are similar to this one. The phase curve through $(-x_0, -y_0)$ is just the image of the phase curve through (x_0, y_0) under a rotation about the origin through π. The phase portrait therefore looks like Figure 3.12.

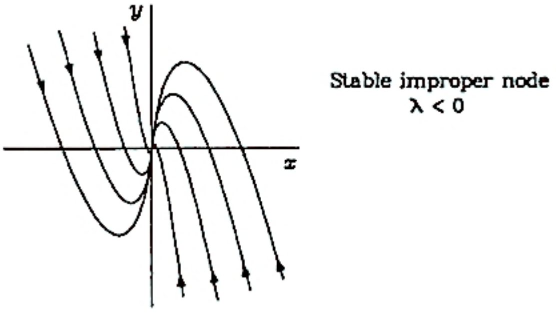


Figure 3.12 Stable improper node, $\lambda < 0$.

When $\lambda < 0$ all points approach the origin along their phase curves, so the origin is a *strongly stable* fixed point. It is called a *stable improper node*.

Stable improper node

If $\lambda > 0$ then the phase curves are obtained by reversing the arrows and reflecting the phase flow shown in the y-axis, to obtain an *unstable improper node*.

Unstable improper node

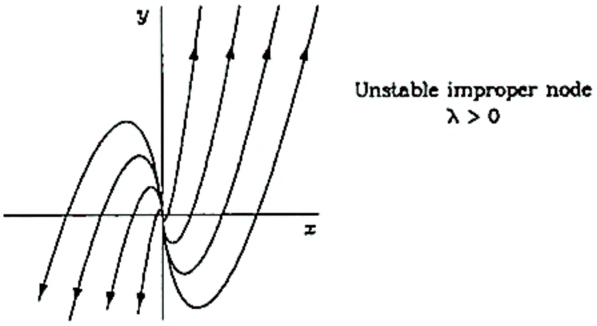


Figure 3.13 Unstable improper node, $\lambda > 0$.

3.5.4 Complex eigenvalues

In this case the eigenvalues are complex conjugates of each other, say $\mu \pm i\nu$, and the coefficient matrix in standard form is given by

$$A = \begin{bmatrix} \mu & \nu \\ -\nu & \mu \end{bmatrix}, \quad \nu \neq 0. \tag{3.33}$$

We solved the equations of motion corresponding to this standard form in *Unit 1*; we shall, however, give an alternative, more elegant, solution here.

Section 1.5, Exercise 1.62

We begin by writing the eigenvalues in terms of their modulus $\rho = |\lambda_1| = |\lambda_2| = \sqrt{\mu^2 + \nu^2}$ and phase ϕ. We have

$$\lambda_1 = \rho e^{i\phi}, \quad \lambda_2 = \rho e^{-i\phi}, \qquad 0 < \phi < \pi, \tag{3.34}$$

so that $\mu = \rho \cos\phi$ and $\nu = \rho \sin\phi$, and the matrix $\mathbf{A}$ becomes

$$\mathbf{A} = \begin{bmatrix} \rho\cos\phi & \rho\sin\phi \\ -\rho\sin\phi & \rho\cos\phi \end{bmatrix} = \rho \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}. \tag{3.35}$$

This represents a rotation through an angle ϕ followed by a uniform scaling, which is a contraction if $\rho < 1$ and an expansion if $\rho > 1$, as shown in Figure 3.14.

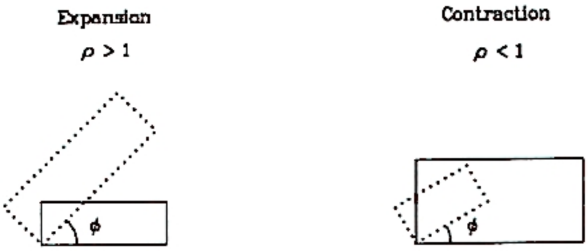


Figure 3.14 The effect of carrying out the transformation corresponding to matrix $\mathbf{A}$ of Equation 3.35 on a rectangle.

This geometric interpretation of the action of $\mathbf{A}$ suggests that the equations of motion would be simpler if we used polar coordinates

$$x = r\cos\theta, \qquad y = r\sin\theta. \tag{3.36}$$

Using the form of $\mathbf{A}$ given in Equation 3.33 the equations of motion $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ are

$$\dot{x} = \mu x + \nu y, \qquad \dot{y} = -\nu x + \mu y. \tag{3.37}$$

We now transform these equations to polar coordinates. Since $r^2 = x^2 + y^2$, differentiation gives

$$\begin{aligned} r\dot{r} &= x\dot{x} + y\dot{y} \\ &= x(\mu x + \nu y) + y(-\nu x + \mu y) \\ &= \mu(x^2 + y^2) \\ &= \mu r^2. \end{aligned} \tag{3.38}$$

Similarly, $\tan\theta = y/x$, so differentiation gives

$$\dot{\theta}\sec^2\theta = \frac{x\dot{y} - y\dot{x}}{x^2} \qquad \text{or} \qquad \dot{\theta} = \frac{x\dot{y} - y\dot{x}}{x^2 + y^2},$$

where we have used the relation $\cos\theta = x/r$. It follows that

$$\dot{\theta} = \frac{x(-\nu x + \mu y) - y(\mu x + \nu y)}{r^2} = -\nu.$$

So we obtain the the simple equations

$$\dot{r} = \mu r, \qquad \dot{\theta} = -\nu. \tag{3.39}$$

The equations of motion are thus separated when expressed in terms of polar coordinates.

The solutions of Equations 3.39 are

$$r(t) = r_0 e^{\mu t}, \qquad \theta = \omega t + \theta_0, \tag{3.40}$$

where we have written ω instead of $-\nu$, for convenience. For $\mu \neq 0$ the phase curves have the polar equations

$$r(\theta) = r_0 \exp\{\mu(\theta - \theta_0)/\omega\}. \tag{3.41}$$

If $\mu < 0$ and $\omega < 0$, then θ decreases steadily and $r \to 0$ exponentially as $t \to \infty$. The phase curves are spirals and the phase portrait for $\mu < 0$, $\omega < 0$, looks like Figure 3.15.

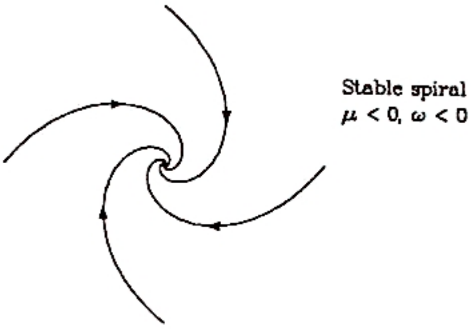


Figure 3.15 Stable spiral, $\mu < 0$, $\omega < 0$.

Changing the sign of ω changes the spiral from a clockwise to an anti-clockwise one, so the phase portrait for $\mu < 0$, $\omega > 0$ looks like Figure 3.16.

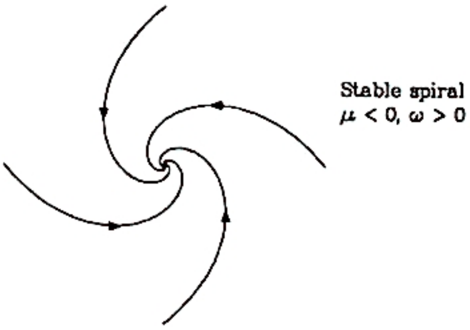


Figure 3.16 Stable spiral, $\mu < 0\ \omega > 0$.

In each case the fixed point is *strongly stable*, and is called a *stable spiral point*.

Stable spiral point

Reversing the sign of μ produces one or the other of these diagrams with the arrows reversed. The fixed point in such a case is *unstable*, and is called an *unstable spiral point*.

Unstable spiral point

If $\mu = 0$ then r is constant, so the phase curves are circles, as in Figure 3.17 below, which is drawn for the case $\omega > 0$.

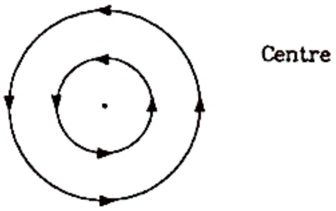


Figure 3.17 Centre.

The fixed point is *stable*, but *not* strongly stable. It is called a *centre*, or sometimes an *elliptic point*.

Centre

Elliptic point

3.5.5 Summary of classification for general linear systems

We have now given specimen phase portraits for all the ten different types of fixed point for simple linear systems. However, these were obtained by assuming that the coefficient matrix was already in standard form; the remaining task, so far as linear systems are concerned, is to describe in a little more detail how the general case is dealt with.

Our method for analysing the general linear system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \tag{3.42}$$

proceeds as follows. We find a new coordinate system $\mathbf{x}' = (x', y')$ in which the equations of motion assume a simpler form. Because the equations of motion, Equation 3.42, are linear we assume that the coordinates $\mathbf{x}$ and $\mathbf{x}'$ are related by a linear, invertible transformation, which we write as

$$\mathbf{x} = \mathbf{P}\mathbf{x}' \qquad \text{or} \qquad \mathbf{x}' = \mathbf{P}^{-1}\mathbf{x}, \tag{3.43}$$

where $\mathbf{P}$ is a constant, non-singular, 2×2 matrix which is to be found. Using the relations

$$\dot{\mathbf{x}} = \mathbf{P}\dot{\mathbf{x}}' \tag{3.44}$$

Equation 3.42 becomes

$$\dot{\mathbf{x}}' = \mathbf{B}\mathbf{x}', \qquad \text{where} \quad \mathbf{B} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}. \tag{3.45}$$

We now appeal to the general theory of 2×2 matrices, outlined in *Unit 1*, according to which for any matrix $\mathbf{A}$ a transformation matrix $\mathbf{P}$ can be found for which $\mathbf{B} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ is in one of the three standard forms. Furthermore, the form of $\mathbf{B}$ is determined by the eigenvalues of $\mathbf{A}$, which we denote by λ_1 and λ_2, and these in turn depend only upon the trace and the determinant of $\mathbf{A}$ through the characteristic equation

Section 1.4

$$\lambda^2 - (\operatorname{tr}\mathbf{A})\lambda + \det\mathbf{A} = 0. \tag{3.46}$$

Equation 3.45 will be formally identical to one of the cases already analysed in the previous subsections; however, the coordinates $\mathbf{x}'$ are not the original Cartesian coordinates but are oblique coordinates related to the Cartesian ones via the matrix $\mathbf{P}$. When the phase portraits of the previous subsections are redrawn to take account of this, the basic shapes of the phase curves remain unchanged; but as $\mathbf{P}$ could be any combination of a rotation, a shear or a dilation, the curves can become somewhat distorted.

We give below a summary of the classification of the fixed points, followed by examples of the phase portraits for each type in the general case. There is no significance in the order in which the types of fixed point are listed.

Suppose that the coefficient matrix $\mathbf{A}$ has eigenvalues λ_1 and λ_2. Then provided that neither λ_1 nor λ_2 is zero, the linear system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ has a single (isolated) fixed point, at the origin. The fixed point is

1. a *stable node* if $\lambda_1 < 0$ and $\lambda_2 < 0$ and $\lambda_1 \neq \lambda_2$;
2. an *unstable node* if $\lambda_1 > 0$ and $\lambda_2 > 0$ and $\lambda_1 \neq \lambda_2$;
3. a *saddle* if λ_1 and λ_2 are real and have different signs;
4. a *stable star* if $\lambda_1 = \lambda_2 = \lambda < 0$ and $\mathbf{A} = \lambda\mathbf{I}$;
5. an *unstable star* if $\lambda_1 = \lambda_2 = \lambda > 0$ and $\mathbf{A} = \lambda\mathbf{I}$;
6. a *stable improper node* if $\lambda_1 = \lambda_2 = \lambda < 0$ and there is only one eigenvector corresponding to λ (up to scalar multiples);
7. an *unstable improper node* if $\lambda_1 = \lambda_2 = \lambda > 0$ and there is only one eigenvector corresponding to λ (up to scalar multiples);
8. a *stable spiral point* if the eigenvalues are complex, $\mu \pm i\nu$, where $\mu < 0$;
9. an *unstable spiral point* if the eigenvalues are complex, $\mu \pm i\nu$, where $\mu > 0$;
10. a *centre* if the eigenvalues are purely imaginary, $\pm i\nu$.

We now provide a pictorial summary of the various types of fixed point.

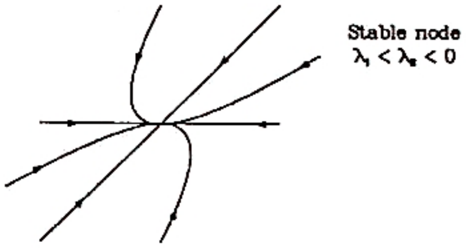

Stable node
$\lambda_1 < \lambda_2 < 0$

Figure 3.18 Stable node, $\lambda_1 < \lambda_2 < 0$.

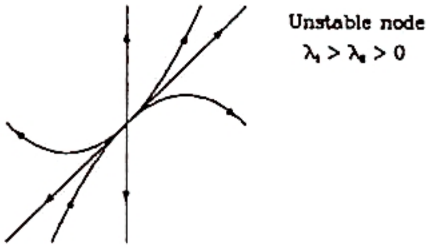

Unstable node
$\lambda_1 > \lambda_2 > 0$

Figure 3.19 Unstable node, $\lambda_1 > \lambda_2 > 0$.

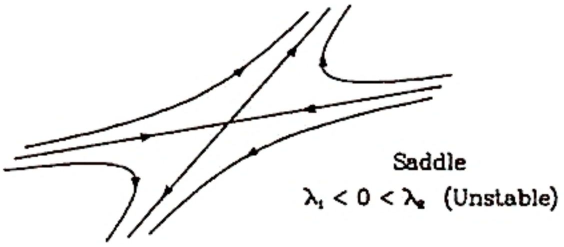

Saddle
$\lambda_1 < 0 < \lambda_2$ (Unstable)

Figure 3.20 Saddle, $\lambda_1 < 0 < \lambda_2$ (unstable).

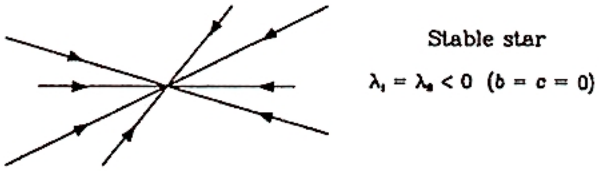

Stable star
$\lambda_1 = \lambda_2 < 0$ $(b = c = 0)$

Figure 3.21 Stable star, $\lambda_1 = \lambda_2 < 0$ $(b = c = 0)$.

The unstable star, $\lambda_1 = \lambda_2 > 0$, $b = c = 0$, is like the stable star but with the arrows reversed.

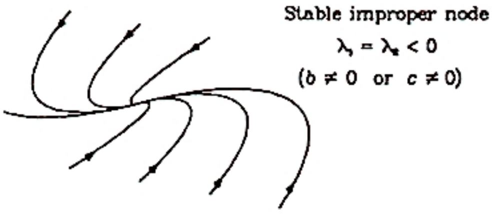

Stable improper node
$\lambda_1 = \lambda_2 < 0$
$(b \neq 0$ or $c \neq 0)$

Figure 3.22 Stable improper node, $\lambda_1 = \lambda_2 < 0$ $(b \neq 0$ or $c \neq 0)$.

The unstable improper node, $\lambda_1 = \lambda_2 > 0$, $b \neq 0$, or $c \neq 0$, is like the stable improper node but with the arrows reversed.

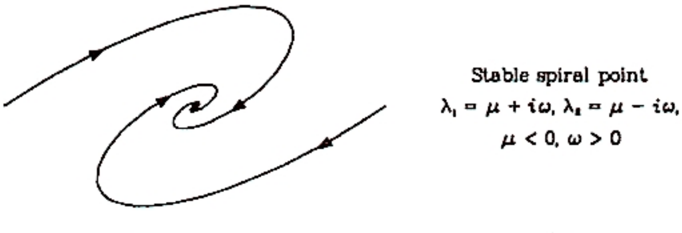

Stable spiral point
$\lambda_1 = \mu + i\omega,\ \lambda_2 = \mu - i\omega,$
$\mu < 0,\ \omega > 0$

Figure 3.23 Stable spiral point, $\lambda_1 = \mu + i\omega$, $\lambda_2 = \mu - i\omega$, $\mu < 0$, $\omega > 0$.

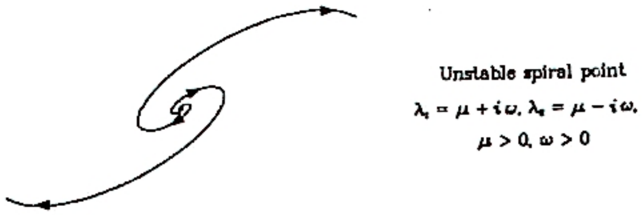


Figure 3.24 Unstable spiral point, $\lambda_1 = \mu + i\omega$, $\lambda_2 = \mu - i\omega$, $\mu > 0$, $\omega > 0$.

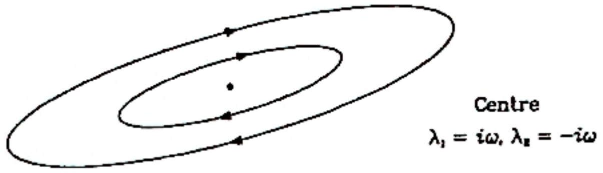


Figure 3.25 Centre, $\lambda_1 = i\omega$, $\lambda_2 = -i\omega$.

Note that the stable fixed points are all strongly stable except for the centre. In fact, if we were concerned only with the stability of fixed points then it would be sufficient to consider a classification into only four distinct types: the strongly stable ones (stable node, star, improper node and spiral); the strongly unstable ones, which are the time-reversals of these; the centre, which is stable but not strongly so; and the saddle, which is unstable but not strongly so. The classification given above is finer than this. The distinctions it makes between the strongly stable cases do correspond to qualitatively different types of motion. For example, the stable node differs from the stable star in that, with two exceptions, its phase curves approach the fixed point tangentially to an eigenline, whereas the phase curves of the stable star are not tangential to each other. Tangency of curves is preserved by any reasonable kind of deformation, and so the stable node and the stable star are qualitatively different.

Exercise 3.25

Classify the fixed points of the following linear systems, and state whether they are unstable, stable or strongly stable.

(i) $\dot{x} = 3x + 4y, \quad \dot{y} = 2x + y$

(ii) $\dot{x} = 3x, \quad \dot{y} = 2x + y$

(iii) $\dot{x} = x + 2y, \quad \dot{y} = -2x + 5y$

(iv) $\dot{x} = x + 2y, \quad \dot{y} = 9x + 4y$

Exercise 3.26

Consider the equations of motion for a damped harmonic oscillator, relative to its equilibrium position, which are

$$\dot{x} = y, \qquad \dot{y} = -\omega^2 x - 2\alpha\omega y,$$

These equations are given in Exercise 3.16, page 54.

where α and ω are constants, with the damping factor α and the natural frequency ω positive. Observe that this is a simple linear system. Examine how the nature of the fixed point depends on the values of α and ω.

Exercise 3.27

Suppose that a real linear system has a coefficient matrix $\mathbf{A}$ which is symmetric, so that $b = c$, where b and c are the off-diagonal entries in $\mathbf{A}$. Investigate the kinds of isolated fixed point such a system can have.

3.6 Fixed points of non-linear systems

When the velocity field $\mathbf{v}(\mathbf{x})$ is given, one of our goals is to sketch the phase flow. The first important step in this direction is to find all the fixed points. Though we cannot usually solve the equations of motion explicitly, we can often find local approximate solutions, which are good approximations over a part of phase space: these are particularly useful in the neighbourhood of fixed points. The approximate solutions are found by replacing the non-linear equations by linear approximations to them, which is the same as the procedure used for first-order systems in *Unit 2*. In most cases the approximate solutions faithfully represent the qualitative features of the true phase curves, even though they are not themselves solutions of the exact equations. This method can provide a sketch of the phase portrait in the neighbourhood of each fixed point, which can sometimes give a good indication of the nature of the whole phase portrait. It follows also that the classification of the isolated fixed points of linear systems developed in the previous section carries over almost entirely to apply to the isolated fixed points of non-linear systems.

Subsection 2.3.2

We discuss these matters in this section, beginning with the problem of identifying the fixed points.

3.6.1 Determination of fixed points

For a linear system there is no difficulty in finding the fixed points: provided the coefficient matrix is non-singular, there is one fixed point. For non-linear systems things are not so straightforward.

The fixed points $\mathbf{x}_\mathrm{f}$ of the system $\dot{\mathbf{x}} = \mathbf{v}(\mathbf{x})$ are at the roots of

$$\mathbf{v}(\mathbf{x}) = \mathbf{0},$$

that is,

$$v_x(x, y) = 0, \qquad v_y(x, y) = 0. \tag{3.47}$$

This is a pair of simultaneous non-linear equations for x and y which have to be solved to find the fixed points. There is no standard method for solving such equations; ingenuity is usually necessary.

A similar problem arises when one is finding the stationary points of a scalar function of two variables.

Example 3.3

Find the fixed points of the system

$$\dot{x} = x^2 + y^2 + 2x, \qquad \dot{y} = xy + y.$$

Solution

We have to solve the equations

$$x^2 + y^2 + 2x = 0,$$
$$(x + 1)y = 0.$$

From the second of these, either $x = -1$ or $y = 0$. When $x = -1$ the first gives $y^2 = 1$, so $y = \pm 1$. When $y = 0$ the first equation becomes

$$x^2 + 2x = 0$$

with solutions $x = 0$ and $x = -2$. There are therefore four fixed points, at

$$(0, 0), \quad (-1, 1), \quad (-1, -1), \quad (-2, 0).$$ ☐

Exercise 3.28

Find the fixed points of each of the following systems.

(i) $\dot{x} = 2xy - 4y - 8, \quad \dot{y} = -x^2 + 4y^2$

(ii) $\dot{x} = -x^2 + y + 2, \quad \dot{y} = 2(x^2 - y^2)$

Exercise 3.29

Explain how to find the fixed points of a non-homogeneous linear system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{c},$$

where $\mathbf{c}$ is a constant vector. Show that the system has a unique fixed point if $\mathbf{A}$ is non-singular, but if $\mathbf{A}$ is singular the system either has no fixed points at all, or its fixed points are not isolated.

Exercise 3.30

Write the following second-order differential equations in standard form, and find their fixed points.

(i) $\ddot{x} - \dot{x} + x^2 - 2x = 0$

(ii) $\ddot{x} - \dot{x}^3 + x + 5 = 0$

Exercise 3.31

Show that the fixed points of the second-order system corresponding to the differential equation

$$\ddot{x} = F(x, \dot{x})$$

all lie on the x-axis of the phase plane, and that their x-coordinates are found by solving the equation

$$F(x, 0) = 0.$$

What is the significance of this result for Newtonian mechanical systems in one dimension?

Exercise 3.32

Show that the second-order autonomous system obtained from the first-order non-autonomous system

$$\dot{x} = v(x, t)$$

has no fixed points.

3.6.2 Linearising the equations near a fixed point

In order to study the motion near a fixed point we linearise the equations of motion, by using a Taylor series expansion around the fixed point $\mathbf{x}_f$. First we must find the first-order Taylor expansions of the components of the velocity field:

$$\begin{aligned} v_x(\mathbf{x}) &= v_x(\mathbf{x}_f) + (x - x_f)\frac{\partial v_x}{\partial x}(\mathbf{x}_f) + (y - y_f)\frac{\partial v_x}{\partial y}(\mathbf{x}_f) \\ &\quad + O\left((\mathbf{x} - \mathbf{x}_f)^2\right), \\ v_y(\mathbf{x}) &= v_y(\mathbf{x}_f) + (x - x_f)\frac{\partial v_y}{\partial x}(\mathbf{x}_f) + (y - y_f)\frac{\partial v_y}{\partial y}(\mathbf{x}_f) \\ &\quad + O\left((\mathbf{x} - \mathbf{x}_f)^2\right). \end{aligned} \tag{3.48}$$

But since $\mathbf{x}_f$ is a fixed point, $\mathbf{v}(\mathbf{x}_f) = \mathbf{0}$. The idea now is to ignore all terms of degree 2 and higher, that is, the terms written $O((\mathbf{x} - \mathbf{x}_f)^2)$ in Equations 3.48. We also move the origin to the fixed point by the change of variables $\mathbf{z} = \mathbf{x} - \mathbf{x}_f$. This leads to the following equations, which we shall take as our linear approximations to the equations of motion:

$$\dot{\mathbf{z}} = \begin{bmatrix} \dfrac{\partial v_x}{\partial x} & \dfrac{\partial v_x}{\partial y} \\ \dfrac{\partial v_y}{\partial x} & \dfrac{\partial v_y}{\partial y} \end{bmatrix} \mathbf{z}, \qquad \mathbf{z} = \mathbf{x} - \mathbf{x}_f, \tag{3.49}$$

where all derivatives are evaluated at $\mathbf{x} = \mathbf{x}_f$, or $\mathbf{z} = \mathbf{0}$. These equations are linear and of the form $\dot{\mathbf{z}} = \mathbf{A}\mathbf{z}$, where the matrix $\mathbf{A}$ is the Jacobian matrix of the velocity field evaluated at the fixed point:

$$\mathbf{A} = \begin{bmatrix} \dfrac{\partial v_x}{\partial x} & \dfrac{\partial v_x}{\partial y} \\ \dfrac{\partial v_y}{\partial x} & \dfrac{\partial v_y}{\partial y} \end{bmatrix}. \tag{3.50}$$

We call these equations the *linearised equations of motion* for the fixed point $\mathbf{x}_f$, and refer to the system with the linearised equations as the *linearised system*. The linearised system has a fixed point at $\mathbf{z} = \mathbf{0}$, corresponding to $\mathbf{x} = \mathbf{x}_f$, the fixed point of the original system. The fixed point of the linearised system will be isolated provided that the coefficient matrix $\mathbf{A}$ is non-singular. In this case we call $\mathbf{x}_f$ a *simple* fixed point of the original system. This definition generalises the idea of simplicity introduced for linear systems on page 57. It also generalises the definition of simplicity of fixed points for first-order non-linear systems, given in *Unit 2*, which is that the derivative of the velocity function should be non-zero at the fixed point. For the second-order system the rule becomes that the determinant of the Jacobian matrix should be non-zero at the fixed point. The derivative for the first-order system is in effect a 1×1 Jacobian matrix. A system for which *all* fixed points are simple we name a *simple system*.

Linearised equations of motion

Linearised system

Simple fixed point

Subsection 2.3.2

Simple system

A simple fixed point of a second-order system is necessarily isolated. But an isolated fixed point of a non-linear system is not necessarily simple, as the example $\mathbf{v}(\mathbf{x}) = (x^2, y^2)$ shows: this has an isolated fixed point at the origin, although the Jacobian matrix of $\mathbf{v}(\mathbf{x})$ at the origin is the zero matrix, which is certainly singular. However, for second-order systems it is easier to discuss simple than non-simple fixed points, just as it was for first-order ones; though that is not to say that it is straightforward for simple fixed points of second-order systems.

The advantage of linearising the equations of motion is that for almost all types of simple fixed points the behaviour of the linear system near the origin gives a good qualitative guide to the behaviour of the original non-linear system near the fixed point $\mathbf{x}_f$ with respect to which one is linearising. We can accordingly use the classification scheme for linear systems developed in the last section as a first step in classifying the simple fixed points of non-linear systems. We adopt the following procedure. First the fixed points of the non-linear system must be found, as described in the previous subsection. Then the Jacobian matrix, Equation 3.50, of the velocity field is evaluated for each fixed point. The result in each case is a numerical matrix $\mathbf{A}$. In practice, if there are several fixed points it is easier to calculate the Jacobian matrix as a function of x and y, and then substitute the values of the fixed points. The eigenvalues, and if necessary eigenvectors, of $\mathbf{A}$ are then found, and the fixed point of the linearised system is classified accordingly; if $\mathbf{A}$ should be singular then one records the fixed point as non-simple. The classification process must be repeated for each fixed point in turn.

Example 3.4

Classify the fixed points of the system

$$\dot{x} = x^2 + y^2 + 2x, \qquad \dot{y} = xy + y.$$

Solution

The fixed points were found in Example 3.3 (page 67). They occur at

$$(0,0), \quad (-1,1), \quad (-1,-1), \quad (-2,0).$$

The Jacobian matrix of the velocity field is

$$\begin{bmatrix} \dfrac{\partial v_x}{\partial x} & \dfrac{\partial v_x}{\partial y} \\ \dfrac{\partial v_y}{\partial x} & \dfrac{\partial v_y}{\partial y} \end{bmatrix} = \begin{bmatrix} 2(x+1) & 2y \\ y & x+1 \end{bmatrix}.$$

At the origin $(0,0)$ the matrix $\mathbf{A}$ for the linearised system is the matrix obtained by substituting $x = 0$ and $y = 0$ in the Jacobian matrix above. Thus

$$\mathbf{A} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}.$$

This is conveniently already diagonal, and its eigenvalues are 2 and 1. The origin is thus an unstable node for the linearised system, and consequently it is called an unstable node for the non-linear system.

At the point $(-1,1)$ the matrix $\mathbf{A}$ for the linearised system is

$$\mathbf{A} = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}.$$

The characteristic equation for this matrix is

$$\lambda^2 - 2 = 0,$$

with roots $\pm\sqrt{2}$. The fixed point of the linear system is thus a saddle, and the point $(-1,1)$ is therefore called a saddle for the non-linear system. ☐

Exercise 3.33

Complete the analysis of the example above.

Exercise 3.34

Classify the fixed points of each of the following systems. (You located their fixed points in Exercise 3.28, page 67.)

(i) $\dot{x} = -4y + 2xy - 8, \quad \dot{y} = 4y^2 - x^2$

(ii) $\dot{x} = y - x^2 + 2, \quad \dot{y} = 2(x^2 - y^2)$

Exercise 3.35

Find the fixed points of the following systems, and classify them.

(i) $\dot{x} = x + 3y + 4, \quad \dot{y} = -6x + 5y - 1$

(ii) $\dot{x} = x + 3y + 1, \quad \dot{y} = -6x + 5y - 1$

(iii) $\dot{x} = 3x + y + 1, \quad \dot{y} = -x + y - 6$

Exercise 3.36

Given any function $f(x,y)$ of two variables, or a scalar field, we can form a second-order system by taking the velocity field to be the gradient of f: thus

$$\mathbf{v} = \operatorname{grad} f$$

or

$$v_x(x,y) = \frac{\partial f}{\partial x}(x,y), \qquad v_y(x,y) = \frac{\partial f}{\partial y}(x,y).$$

Show that the fixed points of this system occur at the stationary points of f. What conditions on the partial derivatives of f ensure that a fixed point is simple? Investigate how the classification of a fixed point is related to whether it is a maximum, a minimum or a saddle point, when considered as a stationary point of f.

The results of Exercise 3.27, page 66, should prove helpful.

Example 3.5

Find and classify the fixed points of the second-order system corresponding to the second-order differential equation

$$\ddot{x} - \dot{x}^2 + x^2 - x = 0.$$

Solution

This example is not in the standard form necessary for the application of the theory, so we use the usual trick and define the new variable $y = \dot{x}$ so that $\ddot{x} = \dot{y}$ and the original equations can be written in the standard form

$$\dot{x} = v_x(x,y) = y, \qquad \dot{y} = v_y(x,y) = y^2 - x^2 + x.$$

The fixed points are located by finding the roots of $v_x = 0$ and $v_y = 0$; these equations give $y = 0$ and $x - x^2 = 0$ or $x = 0, 1$, and so the fixed points are at $(0,0)$ and $(1,0)$. The Jacobian matrix is

$$\begin{bmatrix} 0 & 1 \\ 1-2x & 2y \end{bmatrix}.$$

At the origin,

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

so that the eigenvalues are $\lambda = \pm 1$; they are real and of different sign so the origin is a saddle.

At the point $(1,0)$,

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

which has eigenvalues $\lambda = \pm i$, so this point is a centre of the linearised system. ☐

Exercise 3.37

Classify the fixed points of the second-order systems corresponding to the following equations. (You found the fixed points in Exercise 3.30, page 68.)

(i) $\ddot{x} - \dot{x} + x^2 - 2x = 0$

(ii) $\ddot{x} - \dot{x}^3 + x + 5 = 0$

In the previous analysis we classified fixed points by linearising the equations of motion and ignoring all terms of $O\left((\mathsf{x} - \mathsf{x}_f)^2\right)$. We have not yet discussed in detail how the nature of the flow of the linearised system is related to that of the non-linear system in the vicinity of the fixed point. In particular, we shall want to know how the stability of the fixed point of the linearised system is related to the stability of the fixed point of the non-linear system with respect to which the system is linearised. On the face of it, the inclusion of the higher-order terms in the velocity field may alter the nature of the fixed point. In most cases, fortunately, there is a close correlation between the nature of the fixed points of the linear and the non-linear system. As an example, see the solution to Exercise 3.33, page 83, Figure S3.8.

However, as the following exercise shows, linearisation does not always preserve the form of the phase flow near a fixed point.

Exercise 3.38

Show that the origin is a centre of the linearisation of the system

$$\dot{x} = y - x(x^2 + y^2), \qquad \dot{y} = -x - y(x^2 + y^2).$$

Show that in polar coordinates, $x = r\cos\theta$, $y = r\sin\theta$, the equations of motion are

$$\dot{r} = -r^3, \qquad \dot{\theta} = -1.$$

Hence deduce that the origin of the non-linear system is a strongly stable fixed point, whereas for the linearised system it is stable but not strongly stable.

Despite this example, all is not lost, as the *linearisation theorem* shows that in most cases the form of the fixed point is unaffected by the inclusion of non-linear terms.

Linearisation Theorem

This theorem relates the phase portrait of a non-linear system in the neighbourhood of a *simple* fixed point to that of its linearisation.

We give only a statement of the theorem; its proof is beyond the scope of this course.

> **Linearisation Theorem.** If the non-linear system
>
> $$\dot{\mathbf{x}} = \mathbf{v}(\mathbf{x})$$
>
> has a simple fixed point at $\mathbf{x}_f$, then there is a neighbourhood of $\mathbf{x}_f$ in which the phase portrait of the system and its linearisation are qualitatively equivalent *provided that the fixed point of the linearised system is not a centre.*
>
> In particular, the fixed point $\mathbf{x}_f$ of the non-linear system is strongly stable whenever the origin is a strongly stable fixed point of the linearised system, and $\mathbf{x}_f$ is unstable whenever the origin is unstable for the linearised system.

Thus except in the one case, where the fixed point of the linearised system is a centre, an analysis of the linearised system is sufficient to determine stability; but when this fixed point is a centre it is not possible to say, on the basis of linearisation, whether or not the original fixed point is stable. In this case the linearisation procedure is only the first step in determining the nature of the flow near the fixed point. It is necessary to investigate the effects of the non-linear terms, and there are no standard methods available for doing this; usually some ingenuity is required.

Moreover, the linearisation theorem shows that in all cases, other than a centre, the phase curves near the fixed point are approximated by the linearised system; of course, if one moves too far away from the fixed point the shapes of the phase curves will be altered by the non-linear terms, and the size of this effect is system dependent. Nevertheless,, the linearisation theorem shows that the terms node (stable or unstable, proper or improper), star and spiral point (stable or unstable), and saddle, when applied to a simple fixed point of any system, linear or non-linear, describe unambiguously the nature of the flow near that fixed point.

Exercise 3.39

Discuss the stability of the fixed points of the systems considered in Exercise 3.37.

There is a clue to the reason why centres are special in the form of the linearised equations of motion in polar coordinates, $\dot{r} = 0$, $\dot{\theta} = \omega$. Small perturbations of a linear system whose fixed point is a centre, such as the addition of non-linear terms to the equation for $\dot{r}$, may radically alter the radial motion; this is exactly what happens in Exercise 3.38.

Centres will turn out to be very important in the remainder of the course, because all fixed points of Hamiltonian systems, dealt with in the following units, are either centres or saddles.

3.7 Limit cycles

For first-order systems all motions tend to fixed points or to infinity. This is not the case for second-order systems. Consider a system separable in polar coordinates, where the radial motion has a fixed point other than at the origin, for instance

$$\begin{aligned} \dot{r} &= \alpha r(r - R), \qquad && R > 0, \\ \dot{\theta} &= \omega, && \omega > 0, \end{aligned}$$

where R, α and ω are constants. The circle $r = R$ is a phase curve, and since the phase points cycle around it indefinitely it is known as a *cycle*. Any phase curve which is closed, in the sense of coming back to meet itself, is called a cycle; to be precise, a cycle is a parametrised phase curve for which the functions $x(t)$ and $y(t)$ are both periodic with the same period.

Cycle

We may treat $r = R$ as a fixed point of the radial motion. In the neighbourhood of $r = R$ the radial motion has the linearised form

$$\dot{s} = \beta s,$$

where $s = r - R$ and $\beta = \alpha R$ is a constant. Thus the motion of the linearised system is given by

$$r = R + (r_0 - R)e^{\beta t}, \qquad \theta = \omega t + \theta_0,$$

and the phase curves of the linearised system by

$$r = R + (r_0 - R)\exp\{\beta(\theta - \theta_0)/\omega\}.$$

For $\beta < 0$, motion starting in the neighbourhood of the cycle approaches it as $t \to \infty$, as shown in the phase diagram in Figure 3.26.

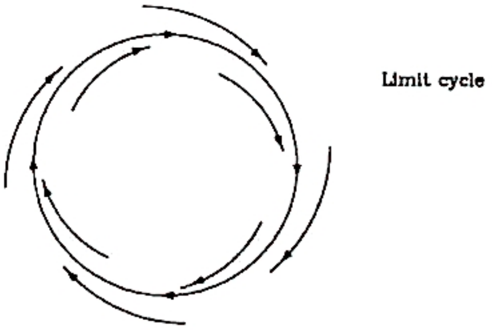


Figure 3.26 Example of a limit cycle which is also an attracting set.

Thus motion on the cycle is stable, in the sense that a phase point initially on the cycle and perturbed slightly returns to it. The cycle is called a *stable limit cycle*. It is an example of an attractor which is a set of points, rather than a single one.

Stable limit cycle

If β is positive, then the arrows point away from the cycle; in this case the cycle is unstable.

Limit cycles are very important for the theory of damped non-linear oscillators. In the example discussed above and in the exercises below, the limit cycles are circular and can be found by a simple coordinate transformation. Most limit cycles are not circular, and the problems of detecting them and of determining their stability are both difficult and important. However, Hamiltonian systems, which are the subject of the remainder of this course, do not possess limit cycles, so we do not consider them in more detail.

Exercise 3.40

Find the limit cycles of the system

$$\dot{x} = y + x(1 - x^2 - y^2), \qquad \dot{y} = -x + y(1 - x^2 - y^2),$$

and discuss their stability.

Exercise 3.41

Classify the fixed points and limit cycles, if any, of the two-dimensional flow defined by the equation

$$(\dot{x}, \dot{y}) = \frac{a}{r^2}\sin(br^2)(x, y) + c(-y, x),$$

where a, b and c are positive constants and $r^2 = x^2 + y^2$.

Solutions to the Exercises in Unit 3

Solution 3.1

The phase velocity is given by

$$\mathbf{v}(x, y) = (y, -g).$$

Solution 3.2

In all cases put $y = \dot{x}$, to obtain

(i) $\dot{x} = y, \quad \dot{y} = -\sin x;$

(ii) $\dot{x} = y, \quad \dot{y} = -bx - 2ay;$

(iii) $\dot{x} = y, \quad \dot{y} = h(t) - g(x) - f(x)y.$

Solution 3.3

In the first case $\dot{x} = y$, so

$$\dot{x} = y, \qquad \dot{y} = x^3 - x - y.$$

In the second case, on putting $y = \dot{x} + x$ and differentiating, we obtain $\dot{y} = \ddot{x} + \dot{x}$ $= x^3 - x$, so

$$\dot{x} = -x + y, \qquad \dot{y} = x^3 - x.$$

Solution 3.4

(i) Since $\dot{y} = \ddot{x} = -x$ the result follows.

(ii) On differentiating the given expressions we obtain

$$\dot{x} = -x_0 \sin t + y_0 \cos t = y,$$
$$\dot{y} = -x_0 \cos t - y_0 \sin t = -x,$$

and since $x(0) = x_0$, $y(0) = y_0$, this is the required solution.

(iii) If r and θ are defined by the equations $x = r\cos\theta$, $y = r\sin\theta$, we have

$$r^2 = x^2 + y^2 \quad \Longrightarrow \quad r\dot{r} = x\dot{x} + y\dot{y},$$
$$\tan\theta = y/x \quad \Longrightarrow \quad \dot{\theta} = (x\dot{y} - y\dot{x})/r^2,$$

and on substituting for $\dot{x}$ and $\dot{y}$ we obtain

$$\dot{r} = 0, \qquad \dot{\theta} = -1.$$

Integration gives

$$r(t) = r_0, \qquad \theta(t) = \theta_0 - t,$$

where r_0 and θ_0 are constants, so that

$$x = r_0 \cos(\theta_0 - t) = x_0 \cos t + y_0 \sin t,$$
$$y = r_0 \sin(\theta_0 - t) = y_0 \cos t - x_0 \sin t,$$

the last equation following because $x_0 = r_0 \cos\theta_0$ and $y_0 = r_0 \sin\theta_0$.

(iv) Since $r = r_0$ and $\theta(t) = \theta_0 - t$, the phase curves are concentric circles traversed in the clockwise direction.

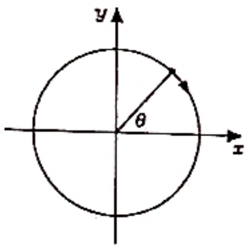


Figure S3.1 A typical phase curve for the equations $\dot{x} = y$, $\dot{y} = -x$.

Solution 3.5

On defining $y = t$ we have $\dot{y} = 1$ and $\dot{x} = v(x, y)$.

Solution 3.6

The solution of

$$\dot{\mathbf{x}} = \mathbf{v}(\mathbf{x}), \qquad \mathbf{x}(t_0) = \mathbf{x}_0,$$

is $\mathbf{x}(t) = \mathbf{x}(\mathbf{x}_0, t_0; t)$. Define $\tilde{\mathbf{x}}(t) = \mathbf{x}(t - t_0)$; then the equation of motion for $\tilde{\mathbf{x}}(t)$ is

$$\frac{d\tilde{\mathbf{x}}}{dt}(t) = \frac{d\mathbf{x}}{dt}(t - t_0) = \mathbf{v}(\mathbf{x}(t - t_0)) = \mathbf{v}(\tilde{\mathbf{x}}(t)), \qquad \text{and} \quad \tilde{\mathbf{x}}(t_0) = \mathbf{x}_0.$$

Thus $\tilde{\mathbf{x}}(t)$ is the unique solution with the given initial condition. So

$$\mathbf{x}(\mathbf{x}_0, t_0; t) = \tilde{\mathbf{x}}(t) = \mathbf{x}(t - t_0) = \mathbf{x}(\mathbf{x}_0, 0; t - t_0),$$

which is a function of $t - t_0$ only.

In general, if $\mathbf{x}(t)$ is any solution and s is any constant, then $\mathbf{x}(t + s)$ is also a solution, by the argument above. Using this result with $s = -(t_1 - t_0)$, we see that $\mathbf{x}(\mathbf{x}_0, t_0; t - (t_1 - t_0))$ is certainly a solution; moreover, on putting $t = t_1$ we obtain

$$\mathbf{x}(\mathbf{x}_0, t_0; t_1 - (t_1 - t_0)) = \mathbf{x}(\mathbf{x}_0, t_0; t_0) = \mathbf{x}_0,$$

which means that

$$\mathbf{x}(\mathbf{x}_0, t_0; t - (t_1 - t_0)) = \mathbf{x}(\mathbf{x}_0, t_1; t).$$

Solution 3.7

It was shown in Solution 3.6 above that for any solution $\mathbf{x}(t)$ and any s, $\mathbf{x}(t + s)$ is also a solution; its initial condition is $\mathbf{x} = \mathbf{x}(s)$ when $t = 0$. Thus

$$\mathbf{x}(\mathbf{x}(\mathbf{x}_0, 0; s), 0; t) = \mathbf{x}(\mathbf{x}_0, 0; t + s) = \mathbf{x}(\mathbf{x}_0, 0; s + t).$$

This is essentially the same as Exercise 2.14 of *Unit 2*.

In terms of the notation for flows (and denoting the general point of phase space by $\mathbf{x}$ rather than $\mathbf{x}_0$), this means that

$$\phi_t(\phi_s(\mathbf{x})) = \phi_{s+t}(\mathbf{x}).$$

Similarly, we have

$$\phi_s(\phi_t(\mathbf{x})) = \phi_{s+t}(\mathbf{x})$$

on reversing the roles of s and t.

Solution 3.8

The solution of the system in Exercise 3.4 with initial condition (x_0, y_0) is

$$x(t) = x_0 \cos t + y_0 \sin t,$$
$$y(t) = -x_0 \sin t + y_0 \cos t.$$

The flow of the system is thus

$$\phi_t(x, y) = (x \cos t + y \sin t, -x \sin t + y \cos t).$$

Thus

$$\begin{aligned}
\phi_s(\phi_t(x, y)) &= \phi_s(x \cos t + y \sin t, -x \sin t + y \cos t) \\
&= ((x \cos t + y \sin t) \cos s + (-x \sin t + y \cos t) \sin s, \\
&\quad - (x \cos t + y \sin t) \sin s + (-x \sin t + y \cos t) \cos s) \\
&= (x(\cos t \cos s - \sin t \sin s) + y(\sin t \cos s + \cos t \sin s), \\
&\quad - x(\cos t \sin s + \sin t \cos s) + y(- \sin t \sin s + \cos t \cos s)) \\
&= (x \cos(s + t) + y \sin(s + t), -x \sin(s + t) + y \cos(s + t)) \\
&= \phi_{s+t}(\mathbf{x}).
\end{aligned}$$

This solution may be shortened by noting that the flow can be written as

$$\phi_t(\mathbf{x}) = \mathbf{A}_t\mathbf{x},$$

where $\mathbf{A}_t$ is the matrix

$$\mathbf{A}_t = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}.$$

To obtain the result it is necessary to show that

$$\mathbf{A}_s\mathbf{A}_t = \mathbf{A}_{s+t},$$

which is a straightforward matter of matrix multiplication, which we asked you to carry out in *Unit 1*. Alternatively, one can note that the matrix $\mathbf{A}_t$ represents a rotation through angle t, and the result says simply that two rotations are combined by adding the angles.

Section 1.4.2, Exercise 1.32

We have asked you to prove this result under the assumption that the definitions and properties of $\cos t$ and $\sin t$ are known beforehand. Another approach, followed in *M203*, is to use the existence theorem to *define* $\cos t$ and $\sin t$ as particular solutions of the given system; then the argument above can be regarded as a deduction of the addition formulae from the general time-translation property of flows.

Block B Unit 4

Solution 3.9

On integrating the equation for $\dot{y}$ we obtain

$$y(t) = y_0 - gt.$$

Since $\dot{x} = y$, integrating this again will give $x(t)$; thus

$$x(t) = x_0 + y_0t - \tfrac{1}{2}gt^2.$$

The flow is given by

$$\phi_t(x,y) = \left(x + yt - \tfrac{1}{2}gt^2, y - gt\right).$$

Thus

$$\begin{aligned}\phi_s(\phi_t(x,y)) &= \phi_s\left(x + yt - \tfrac{1}{2}gt^2, y - gt\right)\\ &= \left(x + yt - \tfrac{1}{2}gt^2 + (y-gt)s - \tfrac{1}{2}gs^2, y - gt - gs\right)\\ &= \left(x + y(s+t) - \tfrac{1}{2}g(s+t)^2, y - g(s+t)\right)\\ &= \phi_{s+t}(x,y).\end{aligned}$$

Solution 3.10

The particle's actual motion is given by $x(t)$, or in other words by the projection of the phase curve onto the x-axis.

Solution 3.11

(i) The velocity field is constant, that is, the phase velocity at each point is the same, and so the phase diagram consists of parallel arrows of the same length.

The phase curves are parallel straight lines of gradient b/a, with equation $a(y - y_0) = b(x - x_0)$, as shown in Figure S3.2.

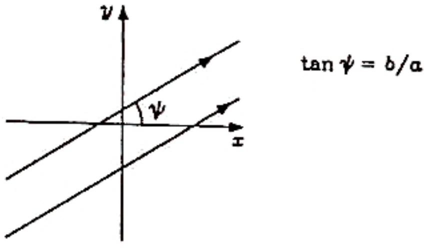


Figure S3.2 Phase curves for the constant velocity field $\mathbf{v} = (a, b)$ with $a, b > 0$.

(ii) The phase diagram is similar to that for the velocity field $\mathbf{v}(x, y) = (y, -g)$ treated in Example 3.1 (page 48) and shown in Figure 3.1 (page 48), except that it is rotated through a right-angle, clockwise if $a > 0$, anti-clockwise if $a < 0$. (In addition, the arrows on the phase curves need to be reversed.)

The phase curves are given by

$$x(t) = x_0 + at, \qquad y(t) = y_0 + x_0 t + \tfrac{1}{2}at^2,$$

or (on eliminating t) by

$$x^2 - 2ay = x_0^2 - 2ay_0 \qquad \text{or} \qquad y = x^2/2a + c,$$

where c is a constant. The last expression is in the standard form for a parabola; c is the value of y at the intersection of the parabola with the y-axis, that is, the y-coordinate of its vertex.

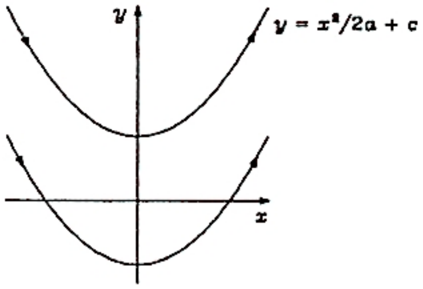


Figure S3.3 Phase curves for the velocity field $\mathbf{v}(\mathbf{x}) = (a, x)$, $a > 0$.

Solution 3.12

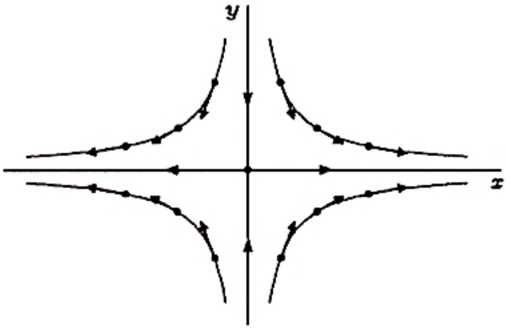


Figure S3.4 Phase diagram for the system with $\mathbf{v}(x, y) = (x, -y)$.

The phase curves are given by

$$x(t) = x_0 e^t, \qquad y(t) = y_0 e^{-t},$$

whose nature is most easily seen by eliminating t, which gives

$$xy = x_0 y_0, \qquad \text{that is,} \quad xy = \text{constant}.$$

For $x_0 y_0 \neq 0$ this is a branch of a rectangular hyperbola, lying completely within the same quadrant as the point (x_0, y_0); the four sections of the coordinate axes are also phase curves, and the origin is a degenerate phase curve.

Solution 3.13

We have

$$\begin{aligned}\frac{dy}{dx} = \frac{y}{x^2} \quad &\Longrightarrow \quad \int_{y_0}^{y} \frac{dy}{y} = \int_{x_0}^{x} \frac{dx}{x^2} \\ &\Longrightarrow \quad \ln y - \ln y_0 = -\frac{1}{x} - \left(-\frac{1}{x_0}\right) \\ &\Longrightarrow \quad y = y_0 e^{1/x_0} e^{-1/x}.\end{aligned}$$

Note that the coordinate axes are phase curves; the sections of the x-axis with $x \neq 0$ are included in the expression above, but the y-axis must be considered separately.

Solution 3.14

We have to solve the equation

$$\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}.$$

On putting $y = xz$, so that $dy/dx = z + x\,dz/dx$, the equation becomes

$$x\frac{dz}{dx} = \frac{(z^2-1)x^2}{2zx^2} - z = -\frac{1+z^2}{2z}.$$

This is solved as follows:

$$\int_{z_0}^{z} dz\,\frac{2z}{1+z^2} = -\int_{x_0}^{x}\frac{dx}{x} \quad\Longrightarrow\quad \ln\left(\frac{1+z^2}{1+z_0^2}\right) = \ln\left(\frac{x_0}{x}\right)$$

$$\Longrightarrow\quad x(1+z^2) = x_0(1+z_0^2).$$

On substituting back for z in terms of y we obtain

$$x_0(x^2+y^2) = x(x_0^2+y_0^2),$$

or, for $x_0 \neq 0$,

$$(x-a)^2 + y^2 = a^2, \qquad \text{where} \quad a = (x_0^2+y_0^2)/2x_0.$$

This is the equation of a circle with its centre at the point $(a, 0)$ and with radius a. The positive and negative y-axes are also phase curves, as is the origin. Note that each circle passes through the origin; however, the origin is a separate phase curve, and so far as the dynamical system is concerned, the circles must be regarded as incomplete, lacking this one point. The phase curves are shown in Figure S3.5.

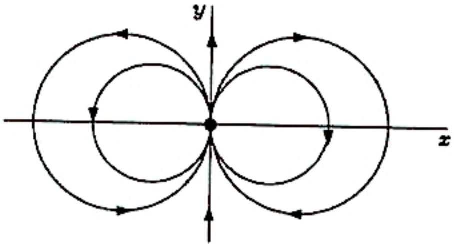


Figure S3.5 Phase curves for the system $\dot{x} = 2xy$, $\dot{y} = y^2 - x^2$.

Solution 3.15

Yes: both the system analysed in Exercise 3.14 and the bead and ring example show that this is possible. From Figure S3.5 it is clear that the origin is simultaneously an attractor and a repellor of the motion on any of the circular phase curves. The top fixed point in the bead and ring example shows similar behaviour. A bead resting at the top fixed point when very slightly disturbed will move away from the top, but will eventually approach the top again but from the other side having passed once round the circle. Thus the top fixed point is also both an attractor and a repellor.

We shall discuss this situation further in *Unit 4*.

Solution 3.16

For all positive α the solution has the form $x(t) = x_f + A(t)e^{-kt}$, where $A(t)$ is either a constant or an oscillating function, according to the relative values of α and ω, and where k is a positive constant. (Note that if $\alpha = 1$, then $A(t)$ is a linear function of t.) The significant term in the solution is the exponentially decaying one. Furthermore, $y(t) = B(t)e^{-kt}$ for some other function $B(t)$, which again is constant or oscillatory. Thus as $t \to \infty$, no matter what the initial conditions are, $(x, y) \to (x_f, 0)$, so the fixed point is strongly stable.

Solution 3.17

The region R_1 contains the motion from the region R_2 if, for every $t > 0$, $\phi_t(R_2)$ is contained in R_1.

Solution 3.18

(i) The three fixed points at A, B and D attract some motion.

(ii) The three fixed points at B, C and D repel some motion.

(iii) The fixed point at A is stable.

(iv) The fixed point at A is strongly stable.

(v) The fixed points at B, C and D are unstable, with C being strongly unstable.

Solution 3.19

We have to show that, given any neighbourhood N of the origin, there is another neighbourhood N' of the origin so that the motion from N' is contained in N. Now the phase curves of the system $\ddot{x} + x = 0$ are circles centred at the origin. There is some positive number ε such that the neighbourhood N contains all points whose distance from the origin is less than ε. Let N' be the set of all points whose distance from the origin is less than ε. Then the motion starting at any point of N' certainly remains within N (it actually remains within N'). So N contains the motion from N'. It follows that the system is stable. But as the motion never approaches the origin it is not strongly stable. The origin is not an attractor.

Solution 3.20

(i) At $\mathbf{x} = 0$, $\dot{\mathbf{x}} = 0$, so the origin is a fixed point.

(ii) On differentiating the given solution we find

$$\dot{x} = x_0 \sinh t + y_0 \cosh t = y,$$
$$\dot{y} = x_0 \cosh t + y_0 \sinh t = x,$$

and as $\cosh(0) = 1$ and $\sinh(0) = 0$, $\mathbf{x}(0) = (x_0, y_0)$.

Alternatively, we can rearrange the equations using the transformation $x = r\cosh\theta$, $y = r\sinh\theta$, to give

$$\dot{x} = \dot{r}\cosh\theta + r\dot{\theta}\sinh\theta = r\sinh\theta,$$
$$\dot{y} = \dot{r}\sinh\theta + r\dot{\theta}\cosh\theta = r\cosh\theta.$$

On using the relation $\cosh^2\theta - \sinh^2\theta = 1$, we find that

$$\dot{r} = 0, \quad \dot{\theta} = 1, \qquad \text{or} \qquad r = r_0, \quad \theta = \theta_0 + t.$$

Thus

$$\begin{aligned} x(t) = r_0\cosh(\theta_0 + t) &= r_0\cosh\theta_0\cosh t + r_0\sinh\theta_0\sinh t \\ &= x_0\cosh t + y_0\sinh t, \\ y(t) = r_0\sinh(\theta_0 + t) &= r_0\sinh\theta_0\cosh t + r_0\cosh\theta_0\sinh t \\ &= y_0\cosh t + x_0\sinh t. \end{aligned}$$

Notice, however, that since $x = r\cosh\theta$ and $y = r\sinh\theta$, then $r^2 = x^2 - y^2 = 0$ if $y = \pm x$. Thus the transformation is defined only on that part of the plane which remains when the lines $y = \pm x$ have been deleted, and the solution beginning at a point on either of these lines must be found by other means. Now from the original equations it follows that $d(x + y)/dt = x + y$ and $d(x - y)/dt = -(x - y)$, so that if $x \pm y$ is ever zero on a solution it is always zero; these lines are themselves phase curves. On the line $y = x$ the equations of motion become $\dot{x} = x$, $\dot{y} = y$, with solution $x(t) = ae^t$, $y(t) = ae^t$, where $a = x_0 = y_0$. On the line $y = -x$, on the other hand, we have $x(t) = ae^{-t}$, $y(t) = -ae^{-t}$, where $a = x_0 = -y_0$.

(iii) Using the definitions $\cosh t = \frac{1}{2}(e^t + e^{-t})$, $\sinh t = \frac{1}{2}(e^t - e^{-t})$, the solution can be written in the form

$$x(t) = \tfrac{1}{2}(x_0 + y_0)e^t + \tfrac{1}{2}(x_0 - y_0)e^{-t},$$
$$y(t) = \tfrac{1}{2}(x_0 + y_0)e^t - \tfrac{1}{2}(x_0 - y_0)e^{-t}.$$

Thus if $y_0 = -x_0$ both $x(t)$ and $y(t)$ behave as e^{-t}, and so go to zero with increasing t. For *any* other (x_0, y_0), the e^t term dominates for large t, so both $x(t)$ and $y(t)$ are unbounded as $t \to \infty$.

Solution 3.21

(i) The fixed points are at $x = 0$, a and b. The graph of $v(x)$ shows that the fixed points at $x = 0, a$ are repellors, and that the one at $x = b$ is an attractor for the points $0 < x < a$. The fixed point at b is strongly stable, those at 0 and a are strongly unstable.

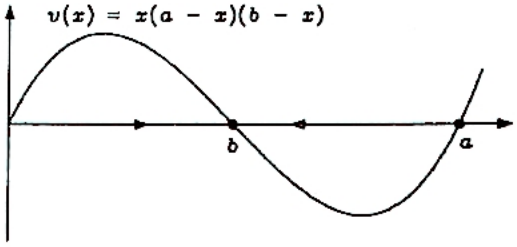


Figure S3.6 Graph of the one-dimensional velocity function $v(x) = x(a - x)(b - x)$.

(ii) The fixed points are at $x^2 = n\pi$, $n = 0, 1, 2, \ldots$. The fixed point at $x = 0$ is special, as near here $\dot{x} \simeq x^2$: it is an attractor for the points $-\pi < x < 0$ and a repellor for the points $0 < x < \pi$. The other fixed points are alternately attractors and repellors, as the graph of $\sin(x^2)$ shows.

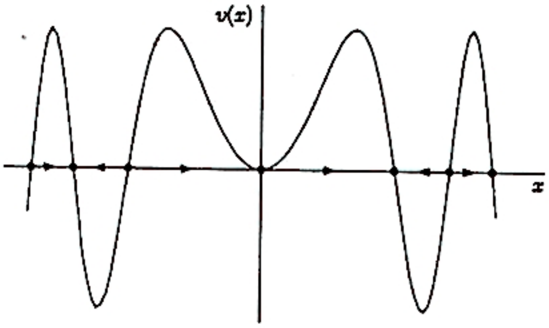


Figure S3.7 Graph of the one-dimensional velocity function $v(x) = \sin(x^2)$.

The fixed points at $x^2 = (2n + 1)\pi$, $n = 0, \pm1, \pm2, \ldots$, are (strongly stable) attractors, and those at $x^2 = 2n\pi$, $n = \pm1, \pm2, \ldots$, are (strongly unstable) repellors. Each attractor attracts all the points in the open interval containing it, bounded by the adjacent fixed points on either side.

Solution 3.22

Each solution of the second-order system is obtained by solving the two first-order equations separately; likewise, the approximate solution is obtained by combining the two separate approximate solutions of the constituent first-order systems. Thus a direct application of the results of *Unit 2* Subsection 2.3.2 leads to the given approximate solution.

If x_f and y_f are both stable, then $b < 0$ and $c < 0$, so that (x_f, y_f) attracts the motion originating at any sufficiently close point (x_0, y_0), and (x_f, y_f) is strongly stable. Similarly, if x_f and y_f are both unstable, then $b > 0$ and $c > 0$; (x_f, y_f) repels the motion originating at any sufficiently close point (x_0, y_0), and (x_f, y_f) is strongly unstable. If one of x_f and y_f is stable and the other unstable, however, so that b and c have opposite signs, then (x_f, y_f) attracts the motion originating from some sufficiently close points (x_0, y_0) but repels that originating from others. In this case (x_f, y_f) is unstable but not strongly so.

Solution 3.23

A fixed point is a solution of $\mathbf{Ax} = \mathbf{0}$. If $\mathbf{A}$ is non-singular then, by definition, the only solution is $\mathbf{x} = \mathbf{0}$. If $\mathbf{A}$ is singular then there is a non-zero vector $\mathbf{x}$ such that $\mathbf{Ax} = \mathbf{0}$; then any scalar multiple of $\mathbf{x}$ also satisfies this equation, and the set of scalar multiples of $\mathbf{x}$ is a radial line through $\mathbf{x}$. To put it another way, if $\mathbf{A}$ is singular then the columns of $\mathbf{A}$ are linearly dependent vectors, which we can take to be scalar multiples $h\mathbf{u}$ and $k\mathbf{u}$ of some fixed vector $\mathbf{u}$. There is then a line of fixed points given by $\mathbf{x} = (-ak, ah)$, since for any a this vector satisfies $\mathbf{Ax} = \mathbf{0}$. Of course, if $\mathbf{A} = \mathbf{0}$ then every point is a fixed point.

Solution 3.24

If $\mathbf{x}(t) = e^{\lambda t}\mathbf{u}$ then $\dot{\mathbf{x}} = \lambda e^{\lambda t}\mathbf{u}$, while $\mathbf{Ax} = e^{\lambda t}\mathbf{Au} = \lambda e^{\lambda t}\mathbf{u}$ since $\mathbf{u}$ is an eigenvector. Thus $\mathbf{x}(t)$ is a solution of the system; moreover, $\mathbf{x}(0) = \mathbf{u}$, so it is the (unique) solution satisfying the given initial condition. As t varies, $\mathbf{x}(t)$ traces out the radial line segment through $\mathbf{u}$, but it never passes through the origin for finite t. If $\lambda < 0$ then $\mathbf{x}(t) \to \mathbf{0}$ as $t \to \infty$, so the origin is an attractor for the motion along the eigenline; while if $\lambda > 0$, $\mathbf{x}(t)$ moves far away from the origin as $t \to \infty$.

Solution 3.25

In all these linear examples we need only find the eigenvalues of the matrix $\mathbf{A}$, where $\dot{\mathbf{x}} = \mathbf{Ax}$; to do so, we need to know only the trace and determinant of $\mathbf{A}$.

(i) $\mathbf{A} = \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}$, $\operatorname{tr}\mathbf{A} = 4$, $\det\mathbf{A} = -5$;

the characteristic equation is

$$\lambda^2 - 4\lambda - 5 = (\lambda - 5)(\lambda + 1) = 0,$$

giving $\lambda = 5, -1$. The eigenvalues are real and of different sign, so the fixed point is a saddle, and is unstable.

(ii) $\mathbf{A} = \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix}$, $\operatorname{tr}\mathbf{A} = 4$, $\det\mathbf{A} = 3$;

thus

$$\lambda^2 - 4\lambda + 3 = (\lambda - 3)(\lambda - 1) = 0,$$

so that $\lambda = 3, 1$. The eigenvalues are real, different and both positive, giving an unstable node.

(iii) $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ -2 & 5 \end{bmatrix}$, $\operatorname{tr}\mathbf{A} = 6$, $\det\mathbf{A} = 9$;

thus

$$\lambda^2 - 6\lambda + 9 = (\lambda - 3)^2 = 0,$$

giving $\lambda = 3$. This is a double positive root; $\mathbf{A}$ is not a multiple of the identity, so the fixed point is an unstable improper node.

(iv) $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 9 & 4 \end{bmatrix}$, $\operatorname{tr}\mathbf{A} = 5$, $\det\mathbf{A} = -14$;

thus

$$\lambda^2 - 5\lambda - 14 = (\lambda - 7)(\lambda + 2) = 0,$$

giving $\lambda = 7, -2$. The eigenvalues are real and of different sign, so the fixed point is a saddle, and is unstable.

Solution 3.26

On writing $y = \dot{x}$ the equations of motion become

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & -2\alpha\omega \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

The trace and determinant of this matrix are $\operatorname{tr}\mathbf{A} = -2\alpha\omega$ and $\det\mathbf{A} = \omega^2$, giving the eigenvalues

$$\lambda = \left(-\alpha \pm \sqrt{\alpha^2 - 1}\right)\omega.$$

So there are three possibilities:

- $\alpha > 1$, eigenvalues real, different and both negative: stable node;
- $\alpha < 1$, eigenvalues complex with negative real part: stable spiral;
- $\alpha = 1$, eigenvalues real, equal and negative: stable improper node.

The case of a stable spiral corresponds to damped oscillatory motion; a stable node gives damped but non-oscillatory motion; a stable improper node corresponds to critical damping.

Solution 3.27

Let the matrix be

$$\mathbf{A} = \begin{bmatrix} a & b \\ b & d \end{bmatrix},$$

so that the characteristic equation is

$$\lambda^2 - (a+d)\lambda + ad - b^2 = 0.$$

Note that

$$(a+d)^2 - 4(ad - b^2) = (a-d)^2 + 4b^2 \geq 0,$$

so the equation will never have complex roots. The eigenvalues are

$$\lambda = \tfrac{1}{2}\left((a+d) \pm \sqrt{(a-d)^2 + 4b^2}\right).$$

The eigenvalues, λ_1 and λ_2, are therefore both real. They will be distinct unless $b = 0$ and $a = d$. Since $\lambda_1\lambda_2 = ad - b^2$ and $\lambda_1 + \lambda_2 = a + d$, if the eigenvalues are distinct the types of fixed point are as follows:

- $ad - b^2 < 0$, eigenvalues of different sign: saddle;
- $ad - b^2 > 0$, $a + d > 0$, eigenvalues both positive: unstable node;
- $ad - b^2 > 0$, $a + d < 0$, eigenvalues both negative: stable node.

When $b = 0$ and $a = d$ the fixed point is a star, stable if $a < 0$, unstable if $a > 0$.

Solution 3.28

(i) The fixed points are at the roots of

$$2xy - 4y - 8 = 0,$$
$$x^2 - 4y^2 = 0.$$

The second equation gives $x = \pm 2y$ which, upon substitution into the first equation, gives

$$\pm y^2 - y - 2 = 0.$$

The positive sign gives $y = -1, 2$ and the negative sign gives no real roots, so the fixed points are at $(-2, -1)$ and $(4, 2)$.

(ii) The fixed points are at the roots of

$$x^2 - y - 2 = 0,$$
$$x^2 - y^2 = 0.$$

The second equation gives $x = \pm y$ which, upon substitution into the first equation, gives

$$x^2 \mp x - 2 = 0.$$

The upper sign gives $x = -1, 2$ and the lower sign gives $x = -2, 1$, so the fixed points are at $(-1, -1)$, $(2, 2)$, $(-2, 2)$ and $(1, -1)$.

Solution 3.29

The fixed points are at $\mathbf{A}\mathbf{x}+\mathbf{c}=\mathbf{0}$ or, if $\mathbf{A}$ is non-singular, $\mathbf{x}=-\mathbf{A}^{-1}\mathbf{c}$. On moving the origin to this fixed point using the transformation $\mathbf{x}=\mathbf{y}-\mathbf{A}^{-1}\mathbf{c}$, the equation of motion becomes $\dot{\mathbf{y}}=\mathbf{A}\mathbf{y}$, which is of the form treated in the text.

If $\mathbf{A}$ is singular then either the equation $\mathbf{A}\mathbf{x}+\mathbf{c}=\mathbf{0}$ has no solution, in which case there are no fixed points, or there is a line of points satisfying the equation, giving a line of non-isolated fixed points.

Solution 3.30

(i) On putting $y=\dot{x}$ the equations become

$$\dot{x}=y, \qquad \dot{y}=y+2x-x^2.$$

At the fixed points $y=0$, whence $2x-x^2=0$ so $x=0$ or 2. The fixed points are therefore $(0,0)$ and $(2,0)$.

(ii) On putting $y=\dot{x}$ the equations become

$$\dot{x}=y, \qquad \dot{y}=y^3-x-5.$$

There is one fixed point, at $(-5,0)$.

Solution 3.31

On putting $y=\dot{x}$ the equations of motion become

$$\dot{x}=y, \qquad \dot{y}=F(x,y),$$

so $y=0$ and $F(x,0)=0$ at the fixed points.

If the differential equation is the equation of motion for a Newtonian mechanical system, then y is the velocity; thus a fixed point of the second-order system is an equilibrium point of the mechanical system in the usual sense, that is, a point at which the system may remain stationary.

Solution 3.32

On putting $y=t$ the equations of motion become $\dot{x}=v(x,y)$, $\dot{y}=1$, and the velocity field is never zero.

Solution 3.33

At the point $(-1,-1)$ the matrix $\mathbf{A}$ for the linearised system is

$$\mathbf{A}=\begin{bmatrix}0 & -2\\ -1 & 0\end{bmatrix},$$

with characteristic equation $\lambda^2-2=0$. The eigenvalues are therefore $\pm\sqrt{2}$, and the fixed point is a saddle of the linearised system (like the one at $(-1,1)$).

At the point $(-2,0)$ the matrix $\mathbf{A}$ for the linearised system is

$$\mathbf{A}=\begin{bmatrix}-2 & 0\\ 0 & -1\end{bmatrix},$$

whose eigenvalues are clearly -2 and -1. The fixed point is therefore a stable node of the linearised system.

It is worth adding to the solution, to show how the fixed point analysis may be used to provide a reasonable guess at the global behaviour of the system. First, we observe that it is helpful to move the origin by the transformation $\xi=x+1$, so that the equations of motion become

We did not expect you to produce this extra work!

$$\dot{\xi}=\xi^2+y^2-1 \qquad (\xi=x+1),$$
$$\dot{y}=y\xi.$$

It should now be clear that the phase curves for $y < 0$ can be obtained from those for $y > 0$ by reflecting them in the ξ-axis, so we need consider only $y \geq 0$. Note also that changing the sign of ξ changes the sign of $\dot{y}$ but not of $\dot{\xi}$; it follows that $dy/d\xi$ is an odd function of ξ, and therefore the equations of the phase curves, $y = f(\xi)$ say, are given by even functions of ξ, $f(-\xi) = f(\xi)$.

In terms of ξ the fixed points are at $y = 0$, $\xi = \pm 1$, and $\xi = 0$, $y = \pm 1$; that is, at the points $(\pm 1, 0)$ and $(0, \pm 1)$. They have been classified as follows.

Fixed point	Type
$(1, 0)$	Unstable node
$(-1, 0)$	Stable node
$(0, \pm 1)$	Saddle

Also notice that on the ξ-axis, $y = 0$, we have $\dot{y} = 0$, so this is a phase curve with the directions shown in the diagram. Also, on the circle $\xi^2 + y^2 = 1$, $\dot{\xi} = 0$, so the phase curves cross this circle parallel to the y-axis in the directions shown. On the y-axis, $\xi = 0$, $\dot{y} = 0$, so the phase curves cross this axis at right-angles in the directions shown.

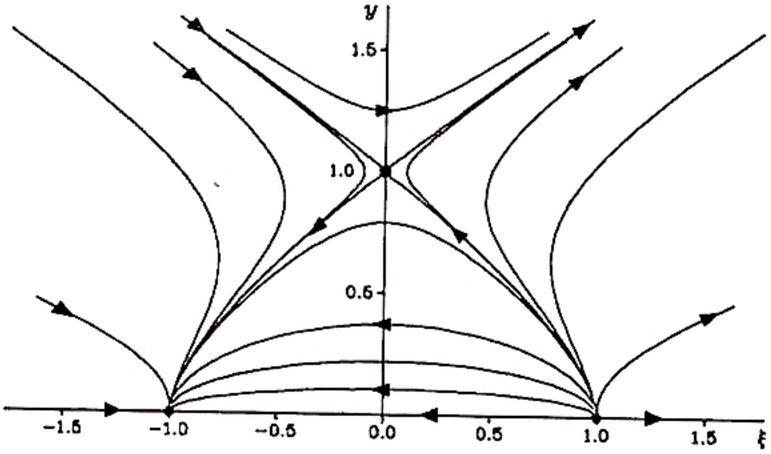


Figure S3.8 Sketch of the phase curves for the system $\dot{\xi} = \xi^2 + y^2 - 1$, $\dot{y} = y\xi$.

All this information is sufficient to guess the whole phase diagram, which is sketched in Figure S3.8.

Further information can be obtained by solving the equations in limiting cases. For example, we shall find an approximation to the phase curve through the point $(0, y_0)$ where $0 \leq y_0 \ll 1$: this will show what the phase curves look like close to the ξ-axis, between the fixed points on it. From the equations of motion we have, for small y,

$$\frac{dy}{d\xi} = \frac{y\xi}{\xi^2 + y^2 - 1} \simeq \frac{y\xi}{\xi^2 - 1}.$$

For $-1 < \xi < 1$ we have, to this approximation,

$$\int_{y_0}^{y} \frac{dy}{y} = -\int_0^{\xi} d\xi \left(\frac{\xi}{1 - \xi^2} \right),$$

or $$\ln\left(\frac{y}{y_0}\right) = \ln\sqrt{1 - \xi^2},$$

or $$\xi^2 + \left(\frac{y}{y_0}\right)^2 = 1.$$

This is the equation of an ellipse joining the two fixed points, which confirms the sketch so far as the region of the phase space close to the ξ-axis is concerned.

Notice how the nature of each fixed point, which we determined from the *linearised* system in each case, is confirmed by the phase portrait for the non-linear system.

On the other hand, if $\xi^2 + y^2 \gg 1$ then

$$\frac{d\xi}{dy} \simeq \frac{\xi^2 + y^2}{\xi y},$$

so on putting $\xi = yu(y)$ we obtain $u du/dy = 1/y$, giving the solution

$$\xi \simeq y\sqrt{C + 2\ln y}, \qquad y \geq 0,$$

for some constant C, with a similar equation for $y < 0$. This gives the shape of the phase curves not shown in the figure. In particular it shows that ξ increases faster than y, contrary to the impression given in the figure, which suggests that ξ is proportional to y; this is one reason why approximate solutions are useful.

Solution 3.34

(i) The matrix $\mathbf{A}$, Equation 3.50 (page 69), is

$$\mathbf{A} = \begin{bmatrix} 2y & 2x - 4 \\ -2x & 8y \end{bmatrix}, \quad \operatorname{tr}\mathbf{A} = 10y, \quad \det\mathbf{A} = 16y^2 + 4x^2 - 8x.$$

The fixed points $\mathbf{x}_f$ are $(-2, -1)$ and $(4, 2)$. They are classified as follows.

$\mathbf{x}_f$	tr $\mathbf{A}$	det $\mathbf{A}$	λ	Fixed point type
$(-2, -1)$	-10	48	$-5 \pm i\sqrt{23}$	Stable spiral
$(4, 2)$	20	96	$8, 12$	Unstable node

(ii) The matrix $\mathbf{A}$ is

$$\mathbf{A} = \begin{bmatrix} -2x & 1 \\ 4x & -4y \end{bmatrix}, \quad \operatorname{tr}\mathbf{A} = -2x - 4y, \quad \det\mathbf{A} = 4x(2y - 1).$$

The fixed points are classified as follows.

$\mathbf{x}_f$	tr $\mathbf{A}$	det $\mathbf{A}$	λ	Fixed point type
$(-1, -1)$	6	12	$3 \pm i\sqrt{3}$	Unstable spiral
$(2, 2)$	-12	24	$-6 \pm \sqrt{12}$	Stable node
$(-2, 2)$	-4	-24	$-2 \pm \sqrt{28}$	Saddle
$(1, -1)$	2	-12	$1 \pm \sqrt{13}$	Saddle

Solution 3.35

(i) The equations $x + 3y + 4 = 0$ and $6x - 5y + 1 = 0$ have solution $(x, y) = (-1, -1)$. The matrix $\mathbf{A}$ is

$$\mathbf{A} = \begin{bmatrix} 1 & 3 \\ -6 & 5 \end{bmatrix}, \quad \operatorname{tr}\mathbf{A} = 6, \quad \det\mathbf{A} = 23,$$

so $\lambda^2 - 6\lambda + 23 = 0$, giving $\lambda = 3 \pm i\sqrt{14}$. The eigenvalues are complex with positive real part, and the fixed point is an unstable spiral.

(ii) The equations $x + 3y + 1 = 0$ and $6x - 5y + 1 = 0$ have solution $(x, y) = (-\frac{8}{23}, -\frac{5}{23})$; but the matrix $\mathbf{A}$ is the same as in part (i), so the fixed point is of the same type.

(iii) The fixed point is at $(x, y) = (-\frac{7}{4}, \frac{17}{4})$, and

$$\mathbf{A} = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix}, \quad \operatorname{tr}\mathbf{A} = 4, \quad \det\mathbf{A} = 4,$$

so $\lambda^2 - 4\lambda + 4 = (\lambda - 2)^2 = 0$, giving $\lambda = 2$. The eigenvalues are real, positive and equal and the fixed point is an unstable improper node.

Solution 3.36

The fixed points are at the points at which

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0,$$

that is, at the stationary points of $f(x,y)$. Linearising about a fixed point gives

$$\mathbf{A} = \begin{bmatrix} A & B \\ B & C \end{bmatrix}, \qquad \text{where} \quad A = \frac{\partial^2 f}{\partial x^2}, \quad B = \frac{\partial^2 f}{\partial x \partial y}, \quad C = \frac{\partial^2 f}{\partial y^2},$$

the second partial derivatives being evaluated at the fixed point. This is precisely the matrix that is used to classify the stationary point. That classification goes as follows:

- $AC - B^2 < 0$: saddle point;
- $AC - B^2 > 0$ and $A < 0$: maximum;
- $AC - B^2 > 0$ and $A > 0$: minimum;
- $AC - B^2 = 0$: no information.

The matrix $\mathbf{A}$ is a symmetric matrix, so the solution to Exercise 3.27 (page 82) applies. The fixed point is simple if

$$\det \mathbf{A} = AC - B^2 \neq 0.$$

From Exercise 3.27 we know that

- if $B \neq 0$, or if $B = 0$ and $A \neq C$, and $AC - B^2 < 0$ then the fixed point will be a saddle;
- if $AC - B^2 > 0$ and $A + C > 0$ it will be an unstable node;
- if $AC - B^2 > 0$ and $A + C < 0$ it will be a stable node;
- if $B = 0$ and $A = C$ then the fixed point will be a stable star if $A < 0$, or an unstable star if $A > 0$.

Thus the fixed point of the dynamical system is a saddle point precisely when the stationary point of f is a saddle point. For a maximum or a minimum we have $AC - B^2 > 0$, so the fixed point is either a node or a star. The condition for a maximum is $A < 0$, for a minimum $A > 0$. Thus if $B = 0$ and $A = C$, the fixed point will be a stable star for a maximum, an unstable star for a minimum. Suppose, finally, that the fixed point is a node. Then since $AC > B^2 \geq 0$, A and C must have the same sign. Thus a maximum will give a stable node, a minimum an unstable node.

(These results may appear to be the wrong way round, since it is usually the case that a minimum of a potential energy function gives stable equilibrium, a maximum unstable equilibrium. Remember, though, that there is a minus sign in the definition of a force in terms of its potential energy function; and in any case, the situation considered in this question cannot be interpreted directly in terms of Newtonian mechanics.)

Solution 3.37

(i) The matrix $\mathbf{A}$ is

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 2(1-x) & 1 \end{bmatrix}, \quad \operatorname{tr} \mathbf{A} = 1, \quad \det \mathbf{A} = 2(x-1),$$

giving the following.

$\mathbf{x}_f$	tr A	det A	λ	Fixed point type
$(0,0)$	1	-2	$-1, 2$	Saddle
$(2,0)$	1	2	$\frac{1}{2}(1 \pm i\sqrt{7})$	Unstable spiral

(ii) The matrix $\mathbf{A}$ is

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 3y^2 \end{bmatrix}, \quad \operatorname{tr} \mathbf{A} = 3y^2, \quad \det \mathbf{A} = 1,$$

giving the following.

$\mathbf{x}_f$	tr **A**	det **A**	λ	Fixed point type
$(-5,0)$	0	1	$\pm i$	Centre

Solution 3.38

The fixed points are at the roots of

$$y - xr^2 = 0, \qquad -x - yr^2 = 0,$$

where $r^2 = x^2 + y^2$, the only solution of which is $\mathbf{x} = (0,0)$. The linearised equations are

$$\dot{x} = y, \qquad \dot{y} = -x,$$

which are treated in Exercise 3.4, where it is shown that the origin is a centre.

Using the results given in the solution to Exercise 3.4 (page 74), we find that for the non-linear system,

$$\dot{r} = -r^3, \qquad \dot{\theta} = -1.$$

This shows that $\dot{r} < 0$ and $\theta = \theta_0 - t$, so that the phase curves spiral into the origin, which is therefore a strongly stable fixed point.

Note that the solution for $r(t)$ is

$$r(t) = \frac{r_0}{\sqrt{1 + 2r_0^2 t}},$$

so that the origin is approached rather more slowly than in the case of the ordinary stable spiral discussed in Subsection 3.5.4, Equation 3.39, where $r(t) = r_0 e^{-|\mu| t}$. This is because $r^3 < r$ for $r < 1$.

Solution 3.39

For the system $\ddot{x} - \dot{x} + x^2 - 2x = 0$ the two fixed points are a saddle and an unstable spiral, neither of which are changed by the inclusion of non-linear terms. But the fixed point of the system $\ddot{x} - \dot{x}^3 + x + 5 = 0$ is a centre of the linearised system, so we cannot conclude that it is a centre of the non-linear system.

In this case put $\xi = x + 5$, $y = \dot{\xi}$, in order to move the fixed point to the origin, so that

$$\dot{\xi} = y, \qquad \dot{y} = -\xi + y^3.$$

Now put $r^2 = \xi^2 + y^2$, so that

$$r\dot{r} = \xi\dot{\xi} + y\dot{y} = \xi y + y(-\xi + y^3) = y^4 > 0.$$

Therefore $\dot{r} > 0$, so the fixed point is an unstable point of the non-linear system; in fact, it is an unstable spiral.

Solution 3.40

The form of these equations suggests using polar coordinates. Using the relations given in the solution to Exercise 3.4 (page 74), we have

$$r\dot{r} = x\left[y + x(1 - r^2)\right] + y\left[-x + y(1 - r^2)\right] = r^2(1 - r^2).$$

Similarly,

$$\dot{\theta} = \frac{x\left[-x + y(1 - r^2)\right] - y\left[y + x(1 - r^2)\right]}{r^2} = -1.$$

The radial equation has a fixed point at $r = 1$, giving a circular limit cycle. On writing $s = r - 1$, we obtain the linear approximation $\dot{s} = -2s$, so that the limit cycle is stable.

Solution 3.41

Again we use polar coordinates, to give

$$r\dot{r} = x\left(\frac{ax}{r^2}\sin br^2 - cy\right) + y\left(\frac{ay}{r^2}\sin br^2 + cx\right) = a\sin br^2,$$

$$r^2\dot{\theta} = x\left(\frac{ay}{r^2}\sin br^2 + cx\right) - y\left(\frac{ax}{r^2}\sin br^2 - cy\right) = cr^2.$$

Thus the limit cycles are at the non-zero roots of

$$\sin br^2 = 0, \qquad \text{that is,} \quad br^2 = n\pi,\ n = 1, 2, \ldots.$$

We determine the stability of the limit cycle at $br^2 = n\pi$ by putting

$$r = \sqrt{\frac{n\pi}{b}} + s,$$

and expanding the radial equation of motion, to obtain

$$\begin{aligned}\dot{s} &= \frac{a}{\sqrt{(n\pi/b)} + s}\sin\left(n\pi + 2s\sqrt{n\pi b} + O(s^2)\right)\\ &= \frac{a}{\sqrt{n\pi/b}}\left(1 - s\sqrt{\frac{b}{n\pi}}\right)\cos n\pi \sin\left(2s\sqrt{n\pi b}\right)\\ &= (2ab\cos n\pi)s + O(s^2)\\ &= (-1)^n 2abs + O(s^2),\end{aligned}$$

where we use the approximation $\sin x = x + O(x^3)$. This shows that the limit cycles alternate in stability, those with odd n being stable, and those with even n being unstable.

There is also a fixed point of the radial equation at $r = 0$; near here

$$(\sin br^2)/r \simeq br,$$

so $\dot{r} = abr$, and the origin is an unstable fixed point.

UNIT 4

INTRODUCTION TO HAMILTONIAN SYSTEMS

4.1 Introduction

It should be clear from *Unit 3* that even if we restrict our attention to systems of order no greater than two, general dynamical systems display such a wealth of possible types of behaviour that it would be difficult to study them all to any depth within the confines of a half-credit course. We shall therefore extract from among all dynamical systems a special class for further study, whose members are known as Hamiltonian systems.

Hamiltonian systems are of especial interest. Historically, this is because systems of particles moving under the influence of conservative forces according to Newton's laws of motion are examples of Hamiltonian systems.

A force is conservative if it can be derived from a potential energy function, as described for one-dimensional systems in *MST204 Unit 4*. The law of conservation of mechanical energy then applies, provided that the force is also independent of the time. Newton's law of universal gravitation, and the derivation of the basic features of planetary motion from it, are discussed in *MST204 Unit 30.*

The most famous example of Newton's work is his treatment of the solar system as a collection of particles moving under the effects of their mutual gravitational attractions. The gravitational force is a conservative, time-independent, force, whose characteristics were also first discovered by Newton and summarised in the inverse square law. Newton was able to explain the basic features of planetary motion, such as the fact that each single planet orbits the Sun in an ellipse. The study of the finer details of planetary motion, which involves taking account of the effects of the other planets as well as the Sun, is a very complicated matter; but nevertheless the Newtonian theory is very successful at predicting the long-term behaviour of planets and in guiding spacecraft through the solar system, as was demonstrated by the remarkable journey of Voyager II.

Hamilton (1805–65) discovered a way of reformulating the Newtonian equations of motion which, whilst not apparently very remarkable in itself, has associated with it a powerful theory which made it a very significant step forward, one whose ramifications are still being explored today. Although the Newtonian theory of the solar system is one of the most important areas of application of Hamilton's theory, it is by no means the only one; modern applications of this theory range from the microscopic motion of electrons in atoms, through molecular motion, to the motion of galaxies. In fact, Hamilton first discovered the equations which bear his name while investigating problems in geometrical optics, the branch of science dealing with the paths taken by light rays travelling through materials with varying optical properties. Any dynamical system, whatever kind of physical system it represents, whose equations of motion can be put into Hamiltonian form as specified below, is called a Hamiltonian system.

Hamiltonian systems form a limited, but very important, class of dynamical systems. Few of the systems that have been discussed in *Units 2* and *3* are Hamiltonian. The specialised theory of Hamiltonian systems is introduced in this unit. We deal only with second-order systems, though the theory in its full generality covers systems of any even order.

We begin by explaining how any Newtonian mechanical system of one degree of freedom – a system consisting of a particle moving along a line or circle, for example – for which the total force is conservative can be treated as a Hamiltonian system; using this case as motivation, we then define what is meant by a Hamiltonian system in general.

4.2 Newtonian and Hamiltonian systems

Consider a Newtonian system consisting of a single particle constrained to move in a one-dimensional space – we call this space the configuration space of the system – and suppose that the particle's position, which is also called the configuration of the system, is determined by a coordinate q. The use of q rather than x is intended to indicate that the configuration space need not be a straight line. Thus q might represent the height of a particle constrained to move vertically; but it might also represent the angle of rotation of a pendulum whose motion is confined to a vertical plane.

The symbol q is used for a similar purpose in *MST204 Unit 7.*

As you know from *Unit 1*, if a dynamical system obeys Newton's laws of motion it is not enough to give its configuration at some particular time to determine the motion for all time; a knowledge of the velocity or momentum is also required. For the Hamiltonian formulation of the equations of motion we take the momentum as the additional variable, and denote it by p. The coordinate q and momentum p define, in two-dimensional phase space, a point (q, p) representing a state of the system; the state at any time t then determines the motion for past and future times. This repeats what we already know: a Newtonian system of one degree of freedom is a dynamical system of second order.

In order to introduce the Hamiltonian formulation in the simplest manner, we specialise to the case of a particle, of constant mass m, moving along a straight line. Let q be the displacement along the line, so that the value of q gives the configuration of the system. Let $p = m\dot{q}$ be the particle's linear momentum. Suppose that the total force on the particle is a function of position but not of velocity (or momentum) or of time, and that it is given by the function $F(q)$. As we showed in *Unit 1*, the equations of motion may be written in the form

$$\dot{q} = \frac{p}{m}, \tag{4.1}$$

$$\dot{p} = F(q). \tag{4.2}$$

The motion of the system is given by the solution of these equations. Because the system has only one degree of freedom, and the force is independent of momentum, it can always be derived from a potential $V(q)$, defined by the equation

$$F(q) = -\frac{dV}{dq}(q), \tag{4.3}$$

where

$$V(q) = -\int_{q_0}^{q} dq\, F(q) \tag{4.4}$$

for some arbitrary q_0. The system is conservative, in the sense that the total mechanical energy $E = \frac{1}{2}m\dot{q}^2 + V(q)$ is conserved, or in other words is constant throughout the motion.

The point to note is that, as a result of the introduction of the potential, the right-hand side of Equation 4.2 has been written as a derivative. Similarly, the right-hand side of Equation 4.1 can be expressed as a derivative:

$$\frac{p}{m} = \frac{d}{dp}\left(\frac{p^2}{2m}\right). \tag{4.5}$$

Notice that the derivatives in Equations 4.3 and 4.5 can be expressed as *partial* derivatives of a *single* function, namely the function $H(q,p)$ given by

$$H(q,p) = \frac{p^2}{2m} + V(q). \tag{4.6}$$

Here q and p are to be treated as *independent* variables for the purposes of partial differentiation. With this understanding, the partial derivatives of H are given by

$$\frac{\partial H}{\partial q} = \frac{\partial V}{\partial q} \quad \text{and} \quad \frac{\partial H}{\partial p} = \frac{\partial}{\partial p}\left(\frac{p^2}{2m}\right) = \frac{p}{m}. \tag{4.7}$$

Then we can write the equations of motion, Equations 4.1 and 4.2, as follows:

$$\dot{q} = \frac{\partial H}{\partial p}(q,p), \qquad \dot{p} = -\frac{\partial H}{\partial q}(q,p). \tag{4.8}$$

These equations are called Hamilton's equations for the Newtonian system. They are equivalent to Newton's equations of motion, but express them in a very special form. The important point is that the components of the phase velocity are determined by the partial derivatives of a single function H. This function should, strictly speaking, be called the Hamiltonian function of the system, but is often simply called its Hamiltonian.

In the example given, the Hamiltonian 4.6 is the sum of the particle's kinetic energy, $p^2/2m$, and its potential energy, $V(q)$. It is generally the case that for the Hamiltonian formulation of a Newtonian dynamical system with conservative forces the Hamiltonian (function) is the total energy. However, there are very many other systems, not arising in this way, which can be expressed in Hamiltonian terms, but for which the Hamiltonian does not have this form; you will meet some later in this course.

Exercise 4.1

By forming Hamilton's equations of motion, confirm that

(i) the Hamiltonian $H(q,p) = p^2/2m$ generates the equations of motion of a free particle, that is, one on which no forces act;

(ii) the Hamiltonian $H(q,p) = p^2/2m + kq$ generates the equations of motion of a particle acted on by a constant force.

Exercise 4.2

Show that the choice of zero level for the potential energy function V does not affect Hamilton's equations, that is, that adding a constant to V leaves the equations unchanged.

Exercise 4.3

Show that the phase velocity, in the Hamiltonian form of the equations of motion, can be obtained by taking the gradient, grad H, of the Hamiltonian function H and rotating it clockwise through a right-angle (assuming that phase space has ordinary Euclidean geometry).

The same approach can be extended to cover the case in which the force may depend on the time, provided that it does not depend on the velocity. Suppose now that the total force on the particle is a function of position and possibly time but not of velocity (or momentum), and that it is given by the function $F(q,t)$. Such a force can always be derived from a potential $V(q,t)$, which is defined by the equation

We must use the partial derivative here, to allow for the possibility that V may depend on t as well as on q.

$$F(q,t) = -\frac{\partial V}{\partial q}(q,t), \tag{4.9}$$

where

$$V(q,t) = -\int_{q_0}^{q} dq\, F(q,t) \tag{4.10}$$

for some arbitrary q_0. This is much the same as before, except for the inclusion of the time. Such a force is said to be conservative, but we shall show in Subsection 4.7.2 that the value of the Hamiltonian is no longer conserved. At first glance this seems confusing, but in dynamics the word 'conservative' is used in two quite different ways. A *force* is said to be conservative if it can be derived from a potential, as in Equation 4.9. A *system* is said to be conservative if there is a quantity which is constant along all phase curves. The overlap between these two uses arises because the energy of a Newtonian system with time-independent, conservative forces is conserved.

Conservative systems are generally easier to understand, so in this unit, apart from this section and Subsection 4.7.2, we shall concentrate on only them. Some non-conservative Hamiltonian systems will be dealt with in later units.

The equations of motion may be derived from a Hamiltonian function $H(q,p,t)$, which is given by

$$H(q,p,t) = \frac{p^2}{2m} + V(q,t). \tag{4.11}$$

We have

$$\frac{\partial H}{\partial q} = \frac{\partial V}{\partial q} \quad \text{and} \quad \frac{\partial H}{\partial p} = \frac{\partial}{\partial p}\left(\frac{p^2}{2m}\right) = \frac{p}{m}. \tag{4.12}$$

The Hamiltonian equations

$$\dot{q} = \frac{\partial H}{\partial p}(q,p,t), \quad \dot{p} = -\frac{\partial H}{\partial q}(q,p,t), \tag{4.13}$$

then give the Newtonian equations of motion, exactly as before.

Exercise 4.4

Show that the Hamiltonian

$$H(q,p,t) = \frac{p^2}{2m} + \tfrac{1}{2}kq^2 - Aq\cos\Omega t$$

leads to the equation of motion for a forced, undamped harmonic oscillator

$$m\ddot{q} + kq = A\cos\Omega t.$$

We have used Newtonian mechanics to introduce the idea of a Hamiltonian system; however, *any* function $H(q,p,t)$ can be used to generate the equations of motion of a second-order dynamical system in the form of Equations 4.13, and our intention in part is to study second-order systems whose equations of motion may be written in this form regardless of how these equations arise. We adopt the same terminology in the more general situation as that used for mechanical systems. A second-order system on a phase space with coordinates q and p is said to be a *Hamiltonian system* if there is a function $H(q,p,t)$ such that

Hamiltonian system

$$\dot{q} = \frac{\partial H}{\partial p}(q,p,t), \quad \dot{p} = -\frac{\partial H}{\partial q}(q,p,t), \tag{4.14}$$

are the equations of motion of the system. The function H is called the *Hamiltonian function*, or simply the *Hamiltonian*, of the system. In all cases, the coordinate q in Equations 4.14 is known as a *generalised coordinate* and the quantity p as its *conjugate momentum*: the pair (q,p) are called *conjugate variables*. However, in general q need not represent the configuration of the system; and even in the case of a mechanical system, p need not be the physical momentum. Because of this generality we shall often (but not always) use (q,p) instead of (x,y) for the coordinates, changing notation from that used in *Unit 3*.

Hamiltonian function, Hamiltonian

Generalised coordinate

Conjugate momentum

Conjugate variables

To summarise: any system whose equations can be cast in the form of Equations 4.14 is named a Hamiltonian system. Besides the conservative mechanical systems which we used to introduce the concept, there are examples of Hamiltonian systems from the theories of electricity, biology, meteorology and economics, for example. It is probably not immediately apparent to you what significance there may be in the fact of being able to write the equations of motion of a dynamical system in the Hamiltonian form; we shall see shortly and in later units that the Hamiltonian structure imposes constraints on the system which considerably reduce the number of possible motions.

There is a test for finding out whether or not a given system is Hamiltonian, which is simple to apply and frequently used; it works as follows. The system with equations of motion

$$\dot{x} = v_x(x, y, t), \qquad \dot{y} = v_y(x, y, t),$$

is Hamiltonian with conjugate variables (x, y) if and only if

$$\frac{\partial v_x}{\partial x} = -\frac{\partial v_y}{\partial y}. \tag{4.15}$$

This assertion may be justified as follows. If the system is Hamiltonian then there exists a function $H(x, y, t)$ such that the equations of motion can be written in the form

$$\dot{x} = v_x(x, y, t) = \frac{\partial H}{\partial y}, \qquad \dot{y} = v_y(x, y, t) = -\frac{\partial H}{\partial x}.$$

From these two equations we can form two expressions for $\partial^2 H/\partial x \partial y$, which must be identical. From the first equation $\partial^2 H/\partial x \partial y = \partial v_x/\partial x$, and from the second, $\partial^2 H/\partial x \partial y = -\partial v_y/\partial y$; thus if the system is Hamiltonian,

$$\frac{\partial v_x}{\partial x} = -\frac{\partial v_y}{\partial y}.$$

Conversely, if this condition is satisfied, then we know that the vector field $(-v_y, v_x)$ is a gradient (with respect to the variables x and y). Thus there is a function $H(x, y, t)$ such that

Unit 1 Subsection 1.2.3

$$v_x(x, y, t) = \frac{\partial H}{\partial y}(x, y, t) \qquad \text{and} \qquad v_y(x, y, t) = -\frac{\partial H}{\partial x}(x, y, t),$$

and the system is Hamiltonian with Hamiltonian function H.

To check whether or not a given second-order dynamical system is Hamiltonian, therefore, one simply has to calculate the appropriate partial derivatives of the components of its velocity field. This is an important result which will frequently be used in later units. We call Equation 4.15 the *mixed derivative rule*, since it depends on the equality of mixed second-order partial derivatives.

Mixed Derivative Rule

Exercise 4.5

Write down Hamilton's equations for the systems with Hamiltonian H given by

(i) $H(q, p) = aq + bp$,
(ii) $H(q, p) = aq^2 + 2bqp + cp^2$,
(iii) $H(q, p) = p \cos q$.

In parts (i) and (ii), a, b and c are constants.

Exercise 4.6

Determine which of the following systems are Hamiltonian, and find a Hamiltonian for those that are.

(i) $\dot{x} = y, \quad \dot{y} = x$
(ii) $\dot{x} = x, \quad \dot{y} = y$
(iii) $\dot{x} = y, \quad \dot{y} = -x$
(iv) $\dot{x} = g(y), \quad \dot{y} = f(x)$
(v) $\dot{x} = yt, \quad \dot{y} = -xt^2$
(vi) $\dot{x} = x - xy, \quad \dot{y} = -y + xy$

In part (iv), $f(x)$ and $g(y)$ are arbitrary functions, respectively of x and y only.

Exercise 4.7

Show that Hamiltonians $H_1(q,p,t)$ and $H_2(q,p,t)$ which differ only by a function of time, so that $H_1(q,p,t) - H_2(q,p,t) = f(t)$, give rise to the same equations of motion.

Exercise 4.8

Many Hamiltonians that arise in physical applications have the dimensions of energy,

$$\mathrm{Dim}(H) = [\mathrm{ML}^2\mathrm{T}^{-2}].$$

Show that, when this is the case, the product of conjugate variables always has the dimensions of *energy* × *time*,

$$\mathrm{Dim}(qp) = [\mathrm{ML}^2\mathrm{T}^{-1}].$$

Hence deduce that if q has the dimensions of an angle, then p has those of an angular momentum.

4.3 Autonomous Hamiltonian systems

The Hamiltonian function may depend on the time, as well as on the phase coordinates. However, for Newtonian systems in which the force is conservative and time-independent, the Hamiltonian is the total energy and is a conserved quantity; the equations of motion are autonomous. In this section we shall examine general autonomous Hamiltonian systems; we shall show that the law of conservation of energy becomes the conservation of the Hamiltonian, and shall discuss some general features of the phase portraits of such conservative systems.

4.3.1 The conservation law

An autonomous Hamiltonian system has a Hamiltonian function, $H(q,p)$, that is independent of time. For an autonomous Hamiltonian system, the value of $H(q,p)$ is constant, or is conserved, along any given motion, or phase curve. To show this we first calculate the derivative of the Hamiltonian with respect to time along any curve $(q(t),p(t))$, using the chain rule, to obtain

$$\frac{dH}{dt} = \frac{\partial H}{\partial q}\frac{dq}{dt} + \frac{\partial H}{\partial p}\frac{dp}{dt}. \tag{4.16}$$

If the curve is a phase curve then Hamilton's equations

$$\frac{dq}{dt} = \dot{q} = \frac{\partial H}{\partial p} \qquad \text{and} \qquad \frac{dp}{dt} = \dot{p} = -\frac{\partial H}{\partial q}$$

are satisfied, and so

$$\frac{dH}{dt} = \frac{\partial H}{\partial q}\frac{\partial H}{\partial p} + \frac{\partial H}{\partial p}\left(-\frac{\partial H}{\partial q}\right) = 0. \tag{4.17}$$

Thus the value of $H(q,p)$ does not change along the phase curve; we say that H is conserved. This result generalises the law of conservation of energy for conservative Newtonian mechanical systems, for which the Hamiltonian is the total energy.

Conservation law for autonomous Hamiltonian systems

Since the conservation law for Hamiltonian systems is a generalisation of the law of conservation of energy for Newtonian systems, it is natural to use similar terminology when discussing it. In the first place, because this conservation law holds, an autonomous Hamiltonian system is often called a *conservative system*. In fact, we shall use the term 'conservative' in the context of Hamiltonian systems to mean only that the value of the Hamiltonian function is constant along all phase curves; thus, unlike the case of Newtonian systems, for Hamiltonian systems the term is unambiguous. Secondly, when dealing with a conservative Hamiltonian system, we shall often call the value that the Hamiltonian function takes along a given motion

Conservative system

the *energy* of the motion. In fact, in this context the term energy is often used as very nearly a synonym for the Hamiltonian function itself.

Energy of a motion of a conservative Hamiltonian system

Exercise 4.9

The system

$$\dot{x} = y, \qquad \dot{y} = -x$$

of Exercise 4.6(iii) is autonomous and Hamiltonian. Solve the equations of motion, and confirm that the Hamiltonian function is constant along any phase curve.

Exercise 4.10

Show that the equations

$$\dot{q} = p, \qquad \dot{p} = -(\omega^2 q + 2\alpha\omega p), \quad \omega \neq 0,$$

are equivalent to the single second-order equation

$$\ddot{q} + 2\alpha\omega\dot{q} + \omega^2 q = 0$$

for the motion of a damped harmonic oscillator (with unit mass) about equilibrium, and that they cannot be expressed in autonomous Hamiltonian form using (q, p) as conjugate variables unless $\alpha = 0$.

Exercise 4.11

The mechanical energy E of a harmonic oscillator with unit mass is given in terms of q, p and the constants which appear in Exercise 4.10 by

$$E = \tfrac{1}{2}p^2 + \tfrac{1}{2}\omega^2 q^2.$$

By using the chain rule, as in Equation 4.16, show that for a damped harmonic oscillator, energy is dissipated at a rate given by

$$\frac{dE}{dt} = -2\alpha\omega p^2.$$

Exercise 4.12

Find Newton's equation of motion for the time-dependent Hamiltonian

$$H(q, p, t) = \frac{p^2}{2m}e^{-2\alpha\omega t} + \tfrac{1}{2}m\omega^2 q^2 e^{2\alpha\omega t}.$$

Exercise 4.13

Show that if the Hamiltonian depends on t then

$$\frac{dH}{dt} = \frac{\partial H}{\partial t};$$

that is, the rate of change of the Hamiltonian along any phase curve is the same as the partial derivative of the Hamiltonian with respect to time. Deduce that a Hamiltonian system which is conservative is necessarily autonomous.

We pointed out earlier, in Exercise 4.3 (page 91), that in the case of a mechanical system the phase velocity is obtained by rotating the gradient of the Hamiltonian function through a right-angle. In particular, the phase velocity is perpendicular in direction to the gradient of the Hamiltonian. This is true in general, as may be seen as follows. The gradient of H is

$$\operatorname{grad} H = \left(\frac{\partial H}{\partial q}, \frac{\partial H}{\partial p}\right), \tag{4.18}$$

while the phase velocity is given by

$$\mathbf{v} = \left(\frac{\partial H}{\partial p}, -\frac{\partial H}{\partial q}\right). \tag{4.19}$$

Thus

$$\mathbf{v} \cdot \operatorname{grad} H = \frac{\partial H}{\partial p}\frac{\partial H}{\partial q} + \left(-\frac{\partial H}{\partial q}\right)\frac{\partial H}{\partial p} = 0. \qquad (4.20)$$

The fact that $\mathbf{v}$ is perpendicular to $\operatorname{grad} H$ is responsible for the conservation law, as may be seen from Equation 4.17. In fact, for any function $F(q,p)$ on phase space, which does not depend explicitly upon the time, differentiation along any phase curve gives

$$\frac{dF}{dt} = \frac{\partial F}{\partial q}\dot{q} + \frac{\partial F}{\partial p}\dot{p} = \mathbf{v} \cdot \operatorname{grad} F,$$

so that when $F = H$ we have $dH/dt = 0$ because $\mathbf{v} \cdot \operatorname{grad} H = 0$.

4.3.2 Phase portraits of conservative systems

The conservation of the Hamiltonian along phase curves is very helpful for drawing the phase portrait. For any suitable constant E, the equation

$$H(q,p) = E$$

defines a curve in phase space. Since the Hamiltonian is conserved, the motion is along such curves of constant H. These curves are called contours of the Hamiltonian. The contours must therefore give the phase curves of the system. Because of these properties, to draw the phase portrait of a Hamiltonian system it is enough to draw a selection of curves of constant H. We normally take a range of values of E equally spaced and sketch the corresponding curves $H(q,p) = E$; then the difference between the values of H on neighbouring curves will be the same throughout the phase plane. These curves are the phase curves of the system; the only missing information is the speed at which a point representing the state of the system traverses its phase curve, and we shall show below how to obtain this.

The reason for using equally-spaced energies given below, on page 97.

It is possible that for certain values of E there are no real values of q and p which satisfy the equation $H(q,p) = E$, which explains why we have used the word 'suitable' above. For other values of E there may be only one point, so that the curve degenerates to a single point; or more generally it may be a collection of discrete points. We allow these possibilities, so the word 'curve' must be understood to include such degenerate cases. Furthermore, a contour may actually consist of several pieces, each of which is a separate phase curve; examples of this will be given later.

This identification of the phase curves with the contours of H makes use of the fact that we are dealing with second-order systems. For higher-order systems it no longer holds; this matter is discussed briefly in Section 4.8.

The above construction, in the second-order case, is analogous to the drawing of a geographical contour map, in which $H(q,p)$ represents the height above sea level of the point (q,p). The phase curves are just the contours (in the geographical sense). The 'high' and 'low' ground can be indicated by $\oplus$ and $\ominus$ signs. The gradient vector $\operatorname{grad} H$ at (q,p) is perpendicular to the contour through the point and directed uphill, that is from low to high ground. The phase velocity vector is obtained by rotating the gradient vector through a right-angle in the clockwise sense. This means that the phase curves are followed by keeping the $\oplus$ signs to the left and the $\ominus$ signs to the right (see Figure 4.1). All information about the motion of a conservative Hamiltonian system is provided by this simplified phase portrait, and there is no need to draw a grid of arrows on the plane as for the general autonomous systems of *Units 2* and *3*. Examples are given in the following sections.

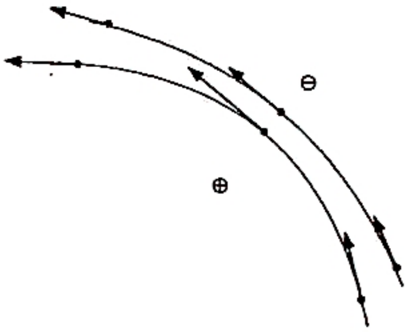


Figure 4.1 Sketch showing that a phase point moves so as to keep the 'high ground' on the left.

Exercise 4.14

Draw phase portraits of the following simple conservative systems.

(i) $H(q,p) = p - q$

(ii) $H(q,p) = \left(p^2 + q^2\right)/2$

Exercise 4.15

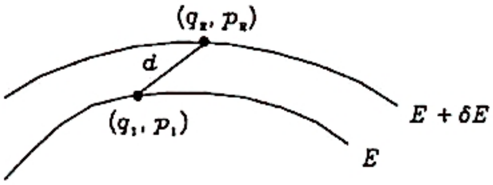


Figure 4.2

Show that the shortest distance, Δ, from the point (q,p), on the contour of energy E, to the contour of energy $E + \delta E$, as shown, is given by

$$\Delta = \frac{\delta E}{|\mathbf{v}|},$$

where $\mathbf{v}$ is the phase velocity. You should assume that δE is small (and ignore terms $O(\delta E^2)$), and that the distance between any two points (q_k, p_k), $k = 1, 2$, in phase space is $d = \sqrt{(q_1 - q_2)^2 + (p_1 - p_2)^2}$.

From Exercise 4.15 it follows that the magnitude of the phase velocity is inversely proportional to the distance between neighbouring contours. This is why it is useful to draw the contours of H at equally-spaced values: one can then see from the contour map how fast the phase point moves along its phase curve. Where the contours (which are the phase curves) are closer together, the phase speed is larger; where they are further apart, the phase speed is smaller.

The equation $H(q,p) = E$ defines the phase curve of energy E of the Hamiltonian $H(q,p)$ implicitly. It is sometimes desirable to be able to represent the curve explicitly, for example as the graph of a function of q, or in other words by solving the equation $H(q,p) = E$ to give p in terms of q (and E). This will always be possible near any point at which the tangent to the curve is not parallel to the p-axis; since the direction of the tangent to the curve is determined by the phase velocity, which is

$$\left(\frac{\partial H}{\partial p}, -\frac{\partial H}{\partial q}\right),$$

we can solve for p over ranges of q for which

$$\frac{\partial H}{\partial p} \neq 0.$$

The solution of the equation $H(q,p) = E$ will usually give several different functions representing separate parts of the phase curve. Consider, for example, the Hamiltonian

$$H(q,p) = q^2 + p^2.$$

For any $E > 0$ we can solve the equation $H(q,p) = E$ for p, to give either

$$p(q,E) = \sqrt{E - q^2}, \qquad q^2 \leq E,$$

or

$$p(q,E) = -\sqrt{E - q^2}, \qquad q^2 \leq E.$$

The contours of H are circles centred at the origin in phase space, and the two explicit formulae for $p(q,E)$ give the upper and the lower semi-circles into which the circle $H(q,p) = E$ is divided by the q-axis. Both of these representations break down in the neighbourhood of either of the points $(\pm\sqrt{E}, 0)$ where the circle crosses the q-axis, as the derivative $\partial p/\partial q$ is infinite where $q = \pm\sqrt{E}$. These are precisely the points at which $\partial H/\partial p = 0$.

In general, when we have solved for p we write the function obtained in the form $p(q,E)$, since different values of E clearly give different curves. We may then treat q and E as independent variables. The function $p(q,E)$ is defined by the fact that

$$H(q, p(q,E)) = E \tag{4.21}$$

identically.

Exercise 4.16

By differentiating Equation 4.21 partially with respect to E and q, and using Hamilton's equations, show that on the phase curve $p(q,E)$ of energy E of the Hamiltonian $H(q,p)$,

(i) $\dot{q}\dfrac{\partial p}{\partial E} = \dfrac{\partial H}{\partial p}\dfrac{\partial p}{\partial E} = 1,$

(ii) $\dot{q}\dfrac{\partial p}{\partial q} = \dot{p}.$

4.3.3 Fixed points and turning points

The fixed points of an autonomous dynamical system with phase velocity $\mathbf{v}$ occur where $\mathbf{v} = \mathbf{0}$, so that by Equations 4.8 (page 91), at a *fixed point* of a conservative Hamiltonian system the gradient of the Hamiltonian is zero:

Fixed point of a conservative Hamiltonian system

$$\left(\frac{\partial H}{\partial q}, \frac{\partial H}{\partial p}\right) = \operatorname{grad} H = \mathbf{0}. \tag{4.22}$$

At the fixed points the system is in equilibrium. The location and nature of the fixed points help us to determine the overall pattern of the motion, as we explained in *Unit 3*. This is illustrated in the following section on linear conservative systems.

Turning points

The points on a phase curve at which $\dot{q} = 0$ but $\dot{p} \neq 0$ are named *turning points*; such points are not as important as fixed points, but knowledge of them helps us to understand the motion and to sketch the phase curves of the system. At a turning point the phase velocity has a zero component in the q direction, and so the tangent to the phase curve there is parallel to the p-axis. This happens when q changes from being increasing to being decreasing as one moves along the curve, or vice versa, as shown in Figure 4.3; hence the name 'turning point'. However, the phase curve may have a vertical inflection at the point (that is, it may cross its vertical tangent), so the terminology is not entirely unambiguous.

For Hamiltonians of the mechanical type, that is, kinetic plus potential energy,

$$H(q,p) = \frac{p^2}{2m} + V(q), \tag{4.23}$$

Hamilton's equation $\dot{q} = \partial H/\partial p = p/m$ shows that all turning points lie on the q-axis. For the phase curve of energy E, the q-coordinates of the turning points are given by the roots of $V(q) = E$. In fact, every point of the q-axis is either a fixed point (if $dV/dq = 0$ there) or a turning point. It follows that for such Hamiltonians the phase curves are perpendicular to the q-axis at the points where they cross it.

There is another way of obtaining this last result. Notice from Equation 4.23 that the Hamiltonian in this case is unchanged by the transformation $(q,p) \mapsto (q,-p)$ of phase space, which corresponds to reflection in the q-axis. It follows that the phase curves are mapped to themselves by this transformation, and are therefore symmetrical with respect to the q-axis. They must therefore cross it perpendicularly.

the case of the Hamiltonian $H(q,p) = q^2 + p^2$ discussed in the previous subsection, the turning points are the points $(\pm\sqrt{E}, 0)$, where the circles of constant H cut the q-axis. They are precisely the points where the representation of the phase curve as a graph breaks down, and this will be true in general, not just in this particular case.

For more general Hamiltonians – those which are not of mechanical type – it will not necessarily be the case that the turning points lie on the q-axis. It will, of course, still be true that at a turning point the direction of the phase curve will be perpendicular to the direction of the q-axis.

The above definition of a turning point picks out the q-coordinate in preference to the p-coordinate. The symmetry of Hamilton's equations suggests that we could equally well define a p-turning point to be a point on a phase curve where $\dot{p} = 0$ but $\dot{q} \neq 0$, as shown in Figure 4.3.

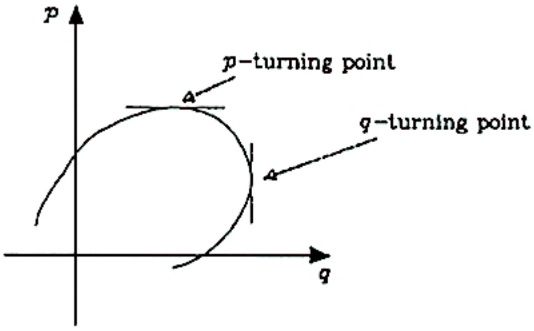


Figure 4.3

Because many Hamiltonians have the form given in Equation 4.23, these turning points are not so important, but for some applications the symmetry between the and the p-coordinates means that both types of turning points need to be considered; for example, to understand the relation between classical and quantum mechanics, both are necessary. Normally, the term 'turning point' is used to mean 'q-turning point'.

4.4 Linear conservative Hamiltonian systems

We shall now exemplify the general ideas concerning phase portraits of conservative Hamiltonians which we introduced in the last section by examining linear systems. The discussion also draws on the principles and methods for the analysis of second-order linear autonomous systems described in *Unit 3* and shows how they specialise the case of linear Hamiltonian systems.

4.4.1 Some examples of linear conservative systems

Recall that by a linear system we mean one whose equations of motion are linear constant-coefficient homogeneous equations.

We shall first present some simple examples of linear behaviour, and then consider the general case. In order to deal with familiar cases, we take our examples from the Newtonian mechanics of a particle of mass m moving in a time-independent potential $V(q)$. The system is then conservative, with Hamiltonian

$$H(q,p) = \frac{p^2}{2m} + V(q), \tag{4.24}$$

from which we obtain Hamilton's equations

$$\dot{q} = \frac{p}{m}, \qquad \dot{p} = -\frac{dV}{dq},$$

which combine to give Newton's equation of motion

$$m\ddot{q} = -\frac{dV}{dq} = F(q), \tag{4.25}$$

where $F(q)$ is the force.

For Hamilton's equations of motion to be linear, the potential $V(q)$ must be quadratic in q. Before we consider the case in which Hamilton's equations are strictly linear, however, we shall discuss a simpler example, the one in which $V(q)$ is linear in q, giving a uniform or constant force $F(q) = -dV/dq =$ constant. In this case Hamilton's equations are linear but not homogeneous.

Example 4.1 **The uniform force field**

The potential is

$$V(q) = aq,$$

where a is constant; we shall assume that $a > 0$. The graph of the potential is shown in Figure 4.4.

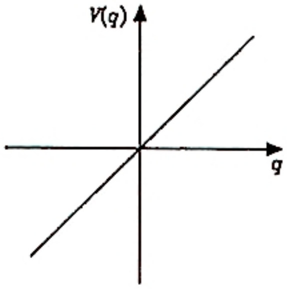


Figure 4.4 Graph of the potential $V(q) = aq$, $a > 0$.

From Equation 4.22 the Hamiltonian is then

$$H(q,p) = \frac{p^2}{2m} + aq,$$

so that the energy contours

$$H(q,p) = E = \text{constant}$$

are the parabolas

$$aq = E - \frac{p^2}{2m}.$$

These are shown in Figure 4.5.

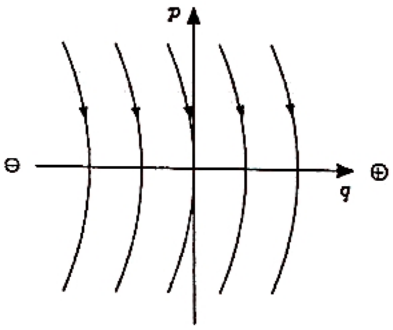


Figure 4.5 Sketch of some contours of the Hamiltonian $H = p^2/2m + aq$, $a > 0$.

There are no fixed points, as

$$\operatorname{grad} H = \left(\frac{\partial H}{\partial q}, \frac{\partial H}{\partial p}\right) = \left(a, \frac{p}{m}\right) \neq 0,$$

but the phase curve of energy E has a turning point at $(E/a, 0)$. Hamilton's equations of motion are simply

$$\dot{q} = p/m, \qquad \dot{p} = -a,$$

leading to Newton's equation

$$m\frac{d^2q}{dt^2} = -a.$$

You will remember that this problem was treated in *Unit 3*, but here we are using the Hamiltonian method to obtain the phase portrait directly, which is more effective than the method used in *Unit 3*. □

Example 3.1, page 48

Example 4.2 The linear oscillator

The Newtonian mechanical system which leads to simple harmonic motion, a system with which you should be very familiar from *MST204 Unit 7*, is strictly linear when viewed as a second-order system. We call the typical corresponding second-order system the linear oscillator: a synonym is 'harmonic' oscillator, which we also use.

If we take the origin of q to be the equilibrium position of the Newtonian system, the force is linear in q and directed towards the origin, $F = -kq$, so the potential is quadratic in q:

$$V(q) = \tfrac{1}{2}kq^2, \qquad k > 0.$$

The graph of the potential is shown in Figure 4.6.

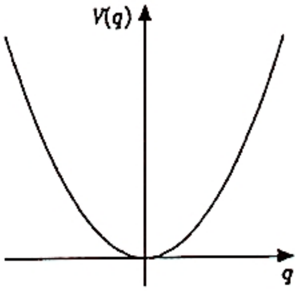


Figure 4.6 Graph of the potential $V(q) = \frac{1}{2}kq^2$, $k > 0$.

The system is conservative, with Hamiltonian

$$H(q,p) = \frac{p^2}{2m} + \tfrac{1}{2}kq^2, \tag{4.26}$$

so that for motion of energy $E > 0$ the contours are given by

$$p^2 + m^2\omega^2q^2 = 2mE, \qquad \omega^2 = \frac{k}{m}. \tag{4.27}$$

These are ellipses, centred at the point $q = p = 0$; the phase portrait is shown in Figure 4.7. The origin is a fixed point, and from the phase portrait it is clear that in terms of the classification given in *Unit 3* it is a centre. It is stable, but not strongly so.

Subsection 3.5.5; see Figure 3.25, page 66.

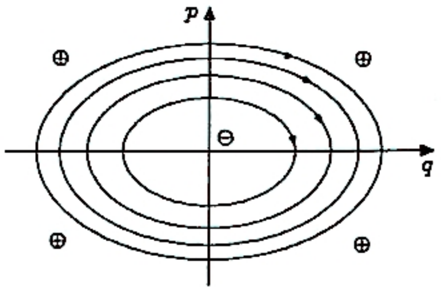


Figure 4.7 Representative equally-spaced energy contours of the Hamiltonian $H = p^2/2m + \frac{1}{2}kq^2$, $k > 0$.

The different elliptical contours, for different values of E, are all similar to each other. If the energy is changed from E to λE, where λ is a positive scaling factor, then the linear dimensions of the ellipse are scaled by the factor $\sqrt{\lambda}$, and the area enclosed by the ellipse is multiplied by the factor λ. As the energy increases, the size of the ellipse increases. Notice that for real contours we need $E \geq 0$, and that the $E = 0$ contour is just the fixed point $q = p = 0$.

The turning points are at $(\pm\sqrt{2E/k}, 0) = (\pm\sqrt{2E/m\omega^2}, 0)$.

Let us examine how the phase portrait of the system is related to the familiar oscillatory motion of the Newtonian system in configuration space. In general terms, the observed motion of the Newtonian system is obtained by looking at the projection onto the q-axis of a point in phase space as it traverses its phase curve. As the phase point circles clockwise round the origin, its projection onto the q-axis oscillates backwards and forwards, as one would expect. To follow this in more detail we solve the equations of motion. Hamilton's equations

$$\dot{q} = \frac{p}{m}, \qquad \dot{p} = -kq$$

lead to the second-order equation

$$\ddot{q} + \omega^2 q = 0, \qquad \omega^2 = k/m,$$

which is, of course, Newton's equation of motion. The familiar general solution of this equation gives the motion

$$q(t) = A\cos(\omega t + \phi), \qquad p(t) = -m\omega A\sin(\omega t + \phi),$$

where the amplitude, A, and the phase, ϕ, are constants. In order to compare this with the equation of the phase curve, Equation 4.27, we write the amplitude in terms of the energy,

$$A = \sqrt{\frac{2E}{m\omega^2}}.$$

Clearly the motion is periodic; its angular frequency is ω, and its period is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}.$$

This all corresponds to the familiar results of the Newtonian analysis of simple harmonic motion.

These results are to be found in *MST204 Unit*

The particle is said to oscillate, or librate, around the fixed point at $q = p = 0$. It is important to notice that its period does *not* depend upon the amplitude of the motion, but only upon the particle's mass, m, and the strength of the potential, k. This property is a peculiarity of linear systems and, as will be seen in later units, makes the behaviour of linear systems atypical.

Note that in the case treated in this example the potential has a minimum, at $q = 0$, corresponding to the fixed point. The minimum value of the potential is zero, and this is also the minimum value of the total energy for which motion is possible. For $E > 0$ motion is confined to the region of configuration space for which $V(q) \leq E$, which corresponds to the segment of the line $V = E$ lying inside the graph of the potential shown in Figure 4.6. Because of its effect in confining the motion, such a potential is called a *potential well*. ☐

Potential well

Example 4.3 The linear repulsive force

We now take the force to be linear in q but directed away from the origin, $F = aq$, so the potential is quadratic but negative,

$$V(q) = -\tfrac{1}{2}aq^2, \qquad a > 0.$$

The graph of this is the reflection in the q-axis of that for the linear oscillator, and shown in Figure 4.8.

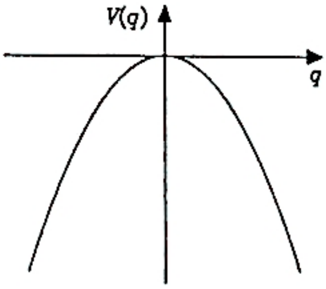


Figure 4.8 Graph of the potential $V(q) = -\frac{1}{2}aq^2$, $a > 0$.

The Hamiltonian is

$$H(q,p) = \frac{p^2}{2m} - \tfrac{1}{2}aq^2, \tag{4.28}$$

so the contour of the motion with energy E has the equation

$$p^2 - m^2\gamma^2q^2 = 2mE, \qquad \gamma^2 = \frac{a}{m}.$$

This is a hyperbola. There is one fixed point, at the origin $q = p = 0$, at which the energy is zero, $E = 0$. As the Hamiltonian factorises,

$$p^2 - m^2\gamma^2q^2 = (p + m\gamma q)(p - m\gamma q),$$

the contour of energy $E = 0$ through the origin consists of the pair of straight lines $= \pm m\gamma q$, shown in Figure 4.9 below. Note that each line segment left when the origin is deleted from this pair of straight lines is a separate phase curve, as is the origin itself (a degenerate phase curve); so the contour actually comprises five phase curves of the same energy. This contour is called a separatrix, for reasons which will be explained shortly.

The fixed point is clearly unstable, and is a saddle as defined in *Unit 3*.

Subsection 3.5.5; see Figure 3.20, page 65.

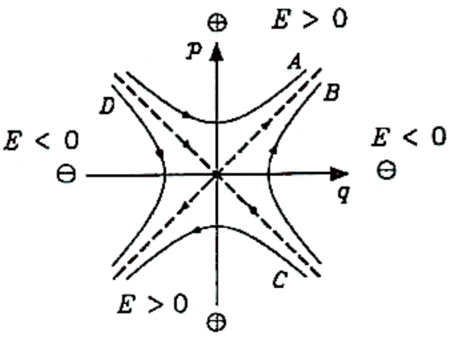


Figure 4.9 Representative contours of the Hamiltonian $H = p^2/2m - \frac{1}{2}aq^2$, $a > 0$. The separatrix is dashed.

In the case of the linear oscillator the energy must be non-negative for motion to be possible. For the linear repulsive force, on the other hand, there is no such restriction on the energy: it may be positive, negative or zero. For $E > 0$, $|p| > m\gamma|q|$, and the corresponding contour is a hyperbola which crosses the p-axis at $p = \pm\sqrt{2mE}$; for $E < 0$, $|q| > |p|/m\gamma$, and the contour is a hyperbola crossing the q-axis at $q = \pm\sqrt{-2mE}/m\gamma$. The hyperbolas are asymptotic to the straight lines $p = \pm m\gamma q$, as shown in Figure 4.9.

In this example we can easily correlate the phase curves with the motion in configuration space. Note that each branch of a hyperbolic contour is a separate phase curve. The potential has a maximum value of 0, which it attains at $q = 0$. No motion with energy $E < 0$ can pass the origin. There is, in effect, a barrier which prevents passage through the origin for motions with insufficient energy. We say that there is a *potential barrier* at the origin. The phase curve D in Figure 4.9 represents motion towards this potential barrier from the left, with subsequent reversal because there is insufficient energy to surmount the barrier. Similarly, the phase curve B (the other branch of the same contour) represents motion towards the barrier from the right, followed by repulsion. In fact, for $E < 0$ motion in configuration space takes place along one or other of the two segments $q > \sqrt{-2E/a}$ and $q < -\sqrt{-2E/a}$.

Potential barrier

When the energy is positive, $E > 0$, the motion can pass over the barrier, with the speed taking its minimum value at $q = 0$; the motion can be either left to right, with $p > 0$, as on phase curve A, or right to left, with $p < 0$, as on phase curve C.

For a fixed point which is a saddle, as the origin is here, there is a contour, or collection of phase curves, for which the fixed point is either an attractor or a repellor. These phase curves are special and are known collectively as a *separatrix* (plural *separatrixes*). It is convenient to use the singular word to label all the attracting or repelling phase curves of a single saddle point, so that a separatrix will normally have several branches. In this example, the separatrix is the set of straight lines $p = \pm m\gamma q$, and so has four distinct branches. It is apparent that the separatrix marks a boundary between phase curves with different properties; it separates the regions of phase space occupied by the phase curves which pass over the potential barrier from the regions of phase space occupied by those which do not. In the next section we shall see further and more dramatic examples of this property of separatrixes, so it should be clear that they are very important phase curves.

Separatrix

So far as motion on the separatrix itself is concerned, each of the four branches must be considered separately. As Figure 4.9 shows, the upper-left branch represents motion with energy $E = 0$ towards the barrier from the left; the lower-right branch represents motion towards the barrier from the right. The upper-right and lower-left branches represent motion from near the barrier to infinity in either direction, again with energy $E = 0$. It is important to remember that, for motion on a phase curve which is attracted by a simple fixed point, the fixed point is not reached in any finite time. This fact was mentioned repeatedly in *Unit 3*; the detailed time-dependence of motion on the separatrix, which will confirm it in this case, is derived in Exercise 4.18 below. Behaviour on the separatrix of a *non-simple* fixed point is, however, different.

The Hamiltonian equations of motion are

$$\dot{q} = \frac{p}{m}, \qquad \dot{p} = aq,$$

leading to the Newtonian equation

$$\ddot{q} - \gamma^2 q = 0.$$

The general solution of the Hamiltonian equations is

$$q(t) = A_+e^{\gamma t} + A_-e^{-\gamma t}, \qquad p(t) = m\gamma\left(A_+e^{\gamma t} - A_-e^{-\gamma t}\right),$$

where A_+ and A_- are constants. Substitution of these solutions into the Hamiltonian gives

$$2mE = m^2\gamma^2\left\{\left(A_+e^{\gamma t} - A_-e^{-\gamma t}\right)^2 - \left(A_+e^{\gamma t} + A_-e^{-\gamma t}\right)^2\right\}$$
$$= -4m^2\gamma^2 A_+A_-,$$

so that

$$E = -2m\gamma^2 A_+A_-.$$

The signs of A_+ and A_- determine the type of motion; on a separatrix, $E = 0$, so either $A_+ = 0$ or $A_- = 0$. ☐

Exercise 4.17

Determine the signs of A_+ and A_- for each of the four phase curves A, B, C and D drawn in Figure 4.9.

Exercise 4.18

Find the motion explicitly on each of the four branches of the separatrix shown in Figure 4.9.

Exercise 4.19

Show that the general solution of Hamilton's equations in this case may also be written in terms of hyperbolic functions; and that on a hyperbolic phase curve for which $A_+A_- > 0$,

$$q(t) = K\cosh(\gamma t + \varepsilon), \qquad p(t) = m\gamma K\sinh(\gamma t + \varepsilon),$$

where K and ε are constants. Relate K and ε to A_+ and A_-, and thence to E. Find an analogous representation for a phase curve for which $A_+A_- < 0$. Investigate the possibility of representing the branches of the separatrix in a similar way.

4.4.2 Linear conservative systems: the general case

We have spent a long time in the last subsection analysing the examples of linear forces, or quadratic potentials. The reason for this is that they exemplify the only types of fixed points possessed by a linear Hamiltonian system, as Exercise 4.20 below shows.

Exercise 4.20

Consider the linear system whose equations of motion are

$$\dot{q} = aq + bp, \qquad \dot{p} = cq + dp.$$

(i) Show that the necessary and sufficient condition for this system to be Hamiltonian is that the trace of the coefficient matrix is zero, that is,

$$a + d = 0.$$

Write down a Hamiltonian function for the system when this condition is satisfied.

(ii) Show that a 2×2 matrix $\mathbf{A}$ for which $\operatorname{tr}\mathbf{A} = 0$ and $\det\mathbf{A} \neq 0$ must have distinct eigenvalues, which may be either real or pure imaginary. Deduce that the fixed point at the origin (assuming it to be isolated) of the second-order linear system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

can be only a saddle or a centre when $\operatorname{tr}\mathbf{A} = 0$. Describe how the nature of the fixed point may be quickly computed from the coefficients a, b and c.

(iii) Show that the two cases of linear forces dealt with in the examples exhaust all the possibilities for the types of fixed point of a linear Hamiltonian system, even though these are not the most general examples of linear Hamiltonian systems.

Thus, compared with general autonomous systems, the possible types of fixed point for a linear Hamiltonian system are very limited. A fixed point of a linear Hamiltonian system cannot be strongly stable or unstable. But the fact that the only stable fixed points are centres may appear to be rather alarming, in view of the use of linearisation in analysing non-linear systems, and the fact that centres comprise the single class of exceptions to the application of the Linearisation Theorem in the general case. We turn next to the consideration of this issue.

Unit 3 Section 3.6, page 71

4.5 Non-linear conservative Hamiltonian systems

We come now to the analysis of non-linear conservative Hamiltonian systems. The first problem to be faced is the difficulty over the Linearisation Theorem which we pointed out at the end of the last section. We shall show in the following subsection that the very nature of the flow of a conservative Hamiltonian system, which we discussed in Section 4.3, makes short work of this problem. When this point is explained, we shall give some examples of non-linear Hamiltonian systems of mechanical type.

4.5.1 Conservative Hamiltonian systems and linearisation

The analysis of the fixed points of a conservative Hamiltonian system involves new features, in comparison with the general case covered in *Unit 3*, which arise just from the fact that the system is Hamiltonian. The first point to make clear is that the linearisation of a Hamiltonian system near a simple fixed point is itself Hamiltonian, that is, it satisfies the criteria given in Exercise 4.20. This is proved in the following exercise.

Exercise 4.21

Suppose that (q_0, p_0) is a simple fixed point of a second-order conservative Hamiltonian system which is not necessarily linear.

Exercise 3.36 of *Unit 3* deals in a similar way with systems in which the velocity field is a gradient, although of course the results are not the same.

(i) Show that (q_0, p_0) is a stationary point of the Hamiltonian function.

(ii) Show that the linearised equations of motion for (q_0, p_0) have the coefficient matrix

$$\begin{bmatrix} \dfrac{\partial^2 H}{\partial q \partial p}(q_0, p_0) & \dfrac{\partial^2 H}{\partial p^2}(q_0, p_0) \\ -\dfrac{\partial^2 H}{\partial q^2}(q_0, p_0) & -\dfrac{\partial^2 H}{\partial q \partial p}(q_0, p_0) \end{bmatrix}.$$

See *Unit 3* Subsection 3.6.2.

Deduce that the linearised system is also Hamiltonian, and that its fixed point must be either a saddle or a centre.

We cannot conclude, from the Linearisation Theorem alone, what type of fixed point a non-linear Hamiltonian system has when the corresponding fixed point of its linearisation is a centre. It is necessary to make further investigations in order to decide, but here the Hamiltonian nature of the system comes to our aid. We know two important facts about the flow of a conservative Hamiltonian system: the phase curves are contours of the Hamiltonian function, and the fixed points are its stationary points. Thus in order to examine the flow near a fixed point of the system, we merely have to consider the behaviour of the contours of the Hamiltonian function near the corresponding stationary point. Imagine a three-dimensional space with coordinates q, p and z. The Hamiltonian function $H(q, p)$ may be represented in this space as the surface with equation $z = H(q, p)$. Its contours may be obtained by slicing through the surface with planes parallel to the (q, p)-plane. Such a plane has equation $z = c =$ constant, and its intersection with the surface is a curve whose projection down onto the (q,p)-plane is the contour $H(q,p) = c$. A stationary point of H is a point where the tangent plane to the surface is horizontal. If the stationary point is a maximum or a minimum then the tangent plane meets the surface there in an isolated point. Nearby, the contours are closed curves, and the corresponding fixed point of the Hamiltonian system is a centre. A three-dimensional plot of a typical minimum is shown in Figure 4.10; in this plot are also shown the closed contour lines, $z =$ constant.

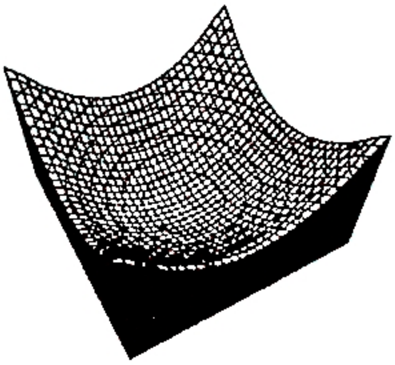


Figure 4.10 Three-dimensional plot of a representative minimum. The circles are the contours.

The only other kind of stationary point is a saddle point, and in this case the tangent plane intersects the surface in a pair of curves which cross at the stationary point. The corresponding fixed point of the dynamical system is clearly a saddle. A three-dimensional plot of a representative saddle is shown in Figure 4.11; the cross shows the contours through the stationary point.

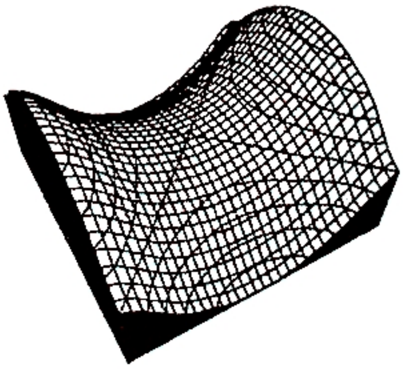


Figure 4.11 Three-dimensional plot of a representative saddle and its contours.

The fact that isolated stationary points of a function of two variables can take only these three possible forms is easy to explain. Consider the intersection of the surface representing the function with a *vertical* plane through the stationary point. The result is a curve which can be thought of as the graph of a function of a single variable. The fact that the point is a stationary point for the original function means that it is also a stationary point for this section function. So generically the graph of the section function will be like a parabola, either going upwards from its vertex, or going downwards. This will be true for all the different section functions as the plane of section rotates, with a few possible exceptions. There are just three possibilities: the parabolic curves all point upwards; or they all point downwards; or some point upwards and some downwards. These possibilities correspond respectively to a minimum, a maximum and a saddle point of the function of two variables.

Since we have exhausted all the possible types of isolated stationary point of the Hamiltonian function, we have shown that the only types of fixed point for a conservative Hamiltonian system are centres and saddles. The Linearisation Theorem holds no further terrors; we can classify the type of a fixed point either by linearisation or by classifying the stationary point of the Hamiltonian into maximum, minimum or saddle point (with the first two corresponding to centres and the last to a saddle). These two procedures are essentially the same, as the following exercise shows.

Exercise 4.22

Suppose that (q_0, p_0) is

(i) a maximum,

(ii) a minimum,

(iii) a saddle point,

of $H(q, p)$. What type is the fixed point of the linearised system in each case?

Exercise 4.23

Show that in the case of a Hamiltonian of mechanical type, where $H(q,p) = p^2/2m + V(q)$, the fixed points correspond to stationary points of the potential function $V(q)$; and correlate the type of fixed point with the nature of the stationary point. Show that the Hamiltonian can never have a maximum in this case.

The main problem in analysing conservative Hamiltonian systems is to determine the location and type of their fixed points; once these have been found, remaining details of the contours are usually relatively easy to establish. The examples treated in the remainder of this section should make this clear.

Before proceeding with the examples, however, it is worth reflecting on the result just obtained. A good example to consider here is the harmonic oscillator, Example 4.2 (page 101). This is a linear conservative Hamiltonian system whose fixed point is a centre. It is a fundamental example of an oscillatory system, and predicts oscillatory motion persisting with the same amplitude forever. But, of course, in practice no real system behaves in this way: because of friction almost any oscillatory system will eventually run down. The damped oscillator gives qualitatively better predictions, therefore; but, as has been shown in Exercise 4.10 (page 95), the equations of motion for this system cannot be put in autonomous Hamiltonian form. In the present context the significant fact about the damped oscillator is that its fixed point is a stable spiral, and therefore strongly stable. This explains, from a different point of view, why it cannot be a conservative Hamiltonian system. In general, friction, which is a common feature of real mechanical systems, leads to strong stability; so systems with friction cannot be conservative Hamiltonian. The dissipation of energy, which is another typical feature of systems with friction, is yet another indicator of the impossibility of giving them conservative Hamiltonian formulation.

This observation naturally raises the question of whether Hamiltonian systems actually exist in nature; the slightest alteration of the equations of motion, by the inclusion of friction for example, will usually destroy the delicate balance which produces the Hamiltonian form. In principle this is true, but in practice nature has conspired to organise itself so that very many natural systems are very well approximated by Hamiltonian equations of motion, so that arbitrary changes which destroy the Hamiltonian nature of the equations of motion are either not allowed or so small that their effect can be seen only over very long times. An example will help to make this clear.

The motion of the Moon is determined primarily by the Sun, with the planets providing perturbations, the perturbation of the Earth being strongest. This motion is described extremely well by Newton's Laws, which can be cast in Hamiltonian form. Tidal effects transform the Moon's orbital motion into heat which is dissipated – mostly, it is thought, in the shallow seas and principally the Bering Strait – causing a slowing of the Earth's rotation by about 2×10^{-3} seconds per century.

This in turn, by the conservation of angular momentum, causes the Moon to move away from the Earth by a few centimetres per year. Such effects are not naturally contained in a Hamiltonian formulation of the equations of motion, but it is clear from the numerical values quoted that they are significant only over very long times.

4.5.2 A cubic potential

From the results of Exercise 4.21 in the previous subsection (page 106), and the discussion which follows it, we know that a simple fixed point of a conservative Hamiltonian system can be of one of only two types: a centre or a saddle. In the case of a Hamiltonian of mechanical type the fixed points occur at points $(q_0, 0)$ for which $V'(q_0) = 0$; a fixed point is simple if and only if $V''(q_0) \neq 0$, and is then a centre or a saddle according as q_0 is at a minimum or a maximum of V.

We now use this analysis of fixed points corresponding to the minima and the maxima of a potential to obtain the qualitative features of the motion for a non-linear Hamiltonian system of mechanical type. The system we shall discuss is that of a particle of unit mass moving in the cubic potential

$$V(q) = \tfrac{1}{2}\omega^2 q^2 - \tfrac{1}{3}Aq^3, \qquad A > 0, \tag{4.29}$$

the special forms of the coefficients in this expression being chosen for convenience. In order to sketch the phase curves, we first sketch the graph of the potential, shown in Figure 4.12, and you should ensure that you understand how Equation 4.29 gives rise to this curve; to this end we now describe some important features of $V(q)$. For $|q|$ large, the term cubic in q dominates, so for q large and positive, $V(q) < 0$, and for q large in magnitude and negative, $V(q) > 0$. At $q = 0$ we have $V(0) = 0$. The stationary points of $V(q)$ are at the roots of

$$\frac{dV}{dq} = \omega^2 q - Aq^2 = 0, \tag{4.30}$$

that is, at $q = 0$ and $q = \omega^2/A$. There is clearly a minimum at $q = 0$ and a maximum at $q = \omega^2/A$. The value of the potential at this local maximum is

$$V\left(\frac{\omega^2}{A}\right) = \frac{\omega^6}{6A^2}. \tag{4.31}$$

When sketching the graph of a potential function like this, it is important only to get the relative positions of the stationary points correct.

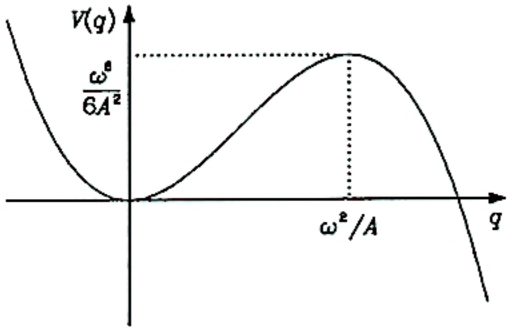


Figure 4.12 Graph of the potential $V(q) = \omega^2 q^2/2 - Aq^3/3$, $A > 0$.

The Hamiltonian of the system is

$$H(q,p) = \tfrac{1}{2}p^2 + V(q) = \tfrac{1}{2}p^2 + \tfrac{1}{2}\omega^2 q^2 - \tfrac{1}{3}Aq^3.$$

Remember that we have taken the particle to have unit mass.

From the graph of the potential we can locate, and classify, the fixed points of the system. They are determined by the stationary points of the potential and are located at $(q,p) = (0,0)$ and $(\omega^2/A, 0)$.

The potential has a minimum at $q = 0$, so the fixed point $(0, 0)$ is a centre, and near here the phase curves are like those shown in Figure 4.7 (page 102). At $q = \omega^2/A$, the potential has a maximum, so the fixed point $(\omega^2/A, 0)$ is a saddle, and near this point the phase curves are like those shown in Figure 4.9 (page 103).

The most important contours to be drawn are the separatrixes, and for a sketch this rarely requires much analysis. A separatrix consists of the phase curves which are attracted to and repelled by a saddle, and in general comprises *some* of the phase curves which have the same energy as the saddle itself. In this example there is only one saddle, and its energy E_s is equal to that of the potential maximum,

$$E_s = \frac{\omega^6}{6A^2}. \qquad (4.32)$$

The equation of the separatrix is therefore

$$p^2 = 2(E_s - V(q)). \qquad (4.33)$$

In sketching the separatrix, however, we have to bear in mind the possibility that this equation describes a curve with several branches, not all of which belong to the separatrix. In particular, we know that the coordinates of the fixed point itself satisfy the equation, and these define a degenerate phase curve which is not strictly speaking part of the separatrix. Equation 4.33 involves only even powers of p, so the contour is symmetrical about the q-axis; indeed, this is true of the whole phase portrait. The contour crosses the q-axis, $p = 0$, when

$$V(q) = E_s. \qquad (4.34)$$

One root of this is, of course, $q = \omega^2/A$, but from the sketch of the potential we see that there is one other, negative, root, at say q_- (which in this simple case is easily seen to be given by $q_- = -\omega^2/2A$). For $q < q_-$ there is no value of p which will satisfy Equation 4.33, since $V(q) > E_s$, and therefore the contour cannot extend to the left of the line $q = q_-$. Near the fixed point $(\omega^2/A, 0)$ the contour will look like the pair of straight lines which cross at the fixed point in the corresponding linear system. For $q > \omega^2/A$ the contour has two branches, corresponding to the two square roots of the right-hand side of Equation 4.33; on the positive branch p is an increasing function of q which tends to $+\infty$ as $q \to \infty$, while on the negative branch it is a decreasing function which tends to $-\infty$ as $q \to \infty$. This information enables us to sketch the contour; the result is shown in Figure 4.13. Note that in this example the whole of the contour, with the exception of the fixed point itself, belongs to the separatrix. The separatrix consists of three branches, each of which is a separate phase curve. The loop at the left corresponds to motion which leaves the vicinity of the potential maximum, moves to the left, bounces off the potential wall at $q_- = -\omega^2/2A$, and returns to the potential maximum, reaching it again only after an infinite time.

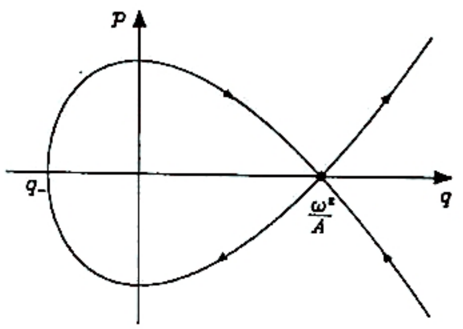


Figure 4.13 Graph showing the separatrix of the cubic potential through the saddle at $(\omega^2/A, 0)$.

For other values of the energy the contour $p(q, E)$ of energy E has the equation

$$p(q, E) = \pm\sqrt{2(E - V(q))}. \qquad (4.35)$$

Thus $p(q, E) = 0$ at the roots of $V(q) = E$; these will give the q-turning points. It is clear from Figures 4.12 and 4.14 (below) that for energies $0 < E < E_s$ the equation $V(q) = E$ has three roots, say $q_k(E)$ with $k = 1, 2, 3$, such that

$$q_1(E) < 0 < q_2(E) < \frac{\omega^2}{A} < q_3(E).$$

Moreover, points (q, p) on the contour of energy E must have either $q_1 \le q \le q_2$ or $q \ge q_3$, since otherwise $E < V(q)$. With this information and the sketch of the separatrix as guidance, it is not difficult to sketch a contour with energy in the given range. The contour falls into two separated parts. The part consisting of points for which $q_1 \le q \le q_2$ is a closed curve lying inside the loop of the separatrix, whilst the part for which $q \ge q_3$ is not a closed curve and lies to the right of the separatrix; it is something like a branch of a hyperbola.

For $E < 0$ the equation $V(q) = E$ has only one root, which is positive; the corresponding contour is in one piece and lacks the closed curve part of the previous case.

For $E > E_s$ the equation $V(q) = E$ has only one root, which is negative. The corresponding contour is again in one piece, but bulges out to the left to include the loop of the separatrix.

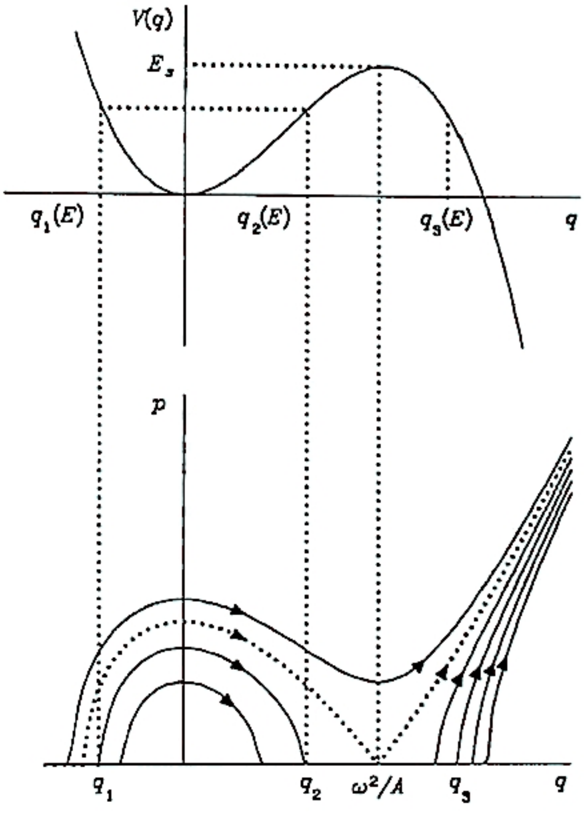


Figure 4.14 The graph in the upper part of the figure is that of the potential $V(q) = \frac{1}{2}\omega^2 q^2 - \frac{1}{3}Aq^3$. The positions of the three turning points on a phase curve of energy E with $E < E_s = \omega^6/6A^2$ are marked. The lower part of the figure shows some contours of the Hamiltonian $H(q, p) = p^2/2 + V(q)$ for $p > 0$; those for $p < 0$ are obtained by reflection in the q-axis.

The upper half of the completed phase portrait is shown in Figure 4.14; the missing half is just the mirror image of the half shown.

From the phase portrait we can see that when the energy satisfies $0 < E < E_s$ bounded, periodic motion is possible. For $E < E_s$, and an initial value of q which is large and positive, motion from the right towards the origin is possible, but with insufficient energy to pass over the potential maximum, so repulsion occurs at $q_3(E) > \omega^2/A$. All motion with $E < 0$ is of this type, and bounded motion is impossible.

For energies $E > E_s$, bounded motion is again impossible; now motion approaching from the right has sufficient energy for the phase point to pass over the top of the potential maximum, and repulsion occurs at $q = q_1(E) < 0$. In this case the contours all lie outside the separatrix. The dip in these contours at $q = \omega^2/A$ shows that the motion is at its slowest when the phase point passes over the potential maximum.

The role of the separatrix should now be clear: it divides phase space into invariant regions, in each of which the motion has a different character. In this case, the loop of the separatrix enclosing the origin contains a region of phase space in which the motion is bounded and periodic. Outside this loop, the motion is unbounded.

Exercise 4.24

Sketch the phase portrait, and describe the possible types of motion, for the system consisting of a particle of unit mass moving under the potential

$$V(q) = -\tfrac{1}{2}\gamma^2 q^2 + \tfrac{1}{3}Bq^3, \qquad B > 0.$$

Exercise 4.25

What is the effect of changing the sign of A in the example and the sign of B in the previous exercise? [Hint: It is the same as the effect of changing the sign of q.]

How are the phase curves for the system with potential

$$V(q) = \tfrac{1}{2}\omega^2 q^2 + \tfrac{1}{3}Aq^3, \qquad A < 0,$$

related to those for the system considered in the example? What about the phase curves for the system with potential

$$V(q) = -\tfrac{1}{2}\gamma^2 q^2 + \tfrac{1}{3}Bq^3, \qquad B < 0?$$

Exercise 4.26

Find how the number and type of the fixed points of the system with cubic potential

$$V(q) = aq + \tfrac{1}{2}bq^2 + \tfrac{1}{3}cq^3, \qquad c \neq 0,$$

depend on the values of a, b and c. Be careful to identify any cases in which there are fixed points which are not simple.

Exercise 4.27

A particle of unit mass moves under the potential

$$V(q) = -\tfrac{1}{2}q^3.$$

It starts at time $t = 0$ at the point $q = 1$ with momentum $p = 1$. Obtain the subsequent motion; show that it terminates, and find the time at which it terminates.

Exercise 4.28

A particle of mass m moves in the potential

$$V(q) = -Aq^\gamma, \qquad A > 0, \quad \gamma > 0.$$

It starts at time $t = 0$ at the point $q = 1$ with momentum $p = \sqrt{2mA}$. For $\gamma \neq 2$, obtain a relation between q and t for $t > 0$ in terms of A, m and $\beta = 1 - \tfrac{1}{2}\gamma$. For which values of A, m and γ does the motion terminate, and for these values at what time does it terminate? What happens if $\gamma = 2$?

4.5.3 A general potential

In general the phase diagram of a conservative Hamiltonian system for a particle moving on the real line has fixed points of both possible types, centres and saddles, corresponding to the minima and the maxima of the potential function $V(q)$. Usually, the motion in a neighbourhood of a fixed point resembles the corresponding linear motion described in Subsection 4.4.1. The saddles are joined by separatrixes that divide the phase space into invariant regions with different types of motion.

These features are illustrated in the example of Figure 4.15, in which a particle moves in the potential $V(q)$. We give no formula for $V(q)$, in order to emphasise that the method is qualitative and that no detailed analysis is required – all we need is the shape of the graph of the potential.

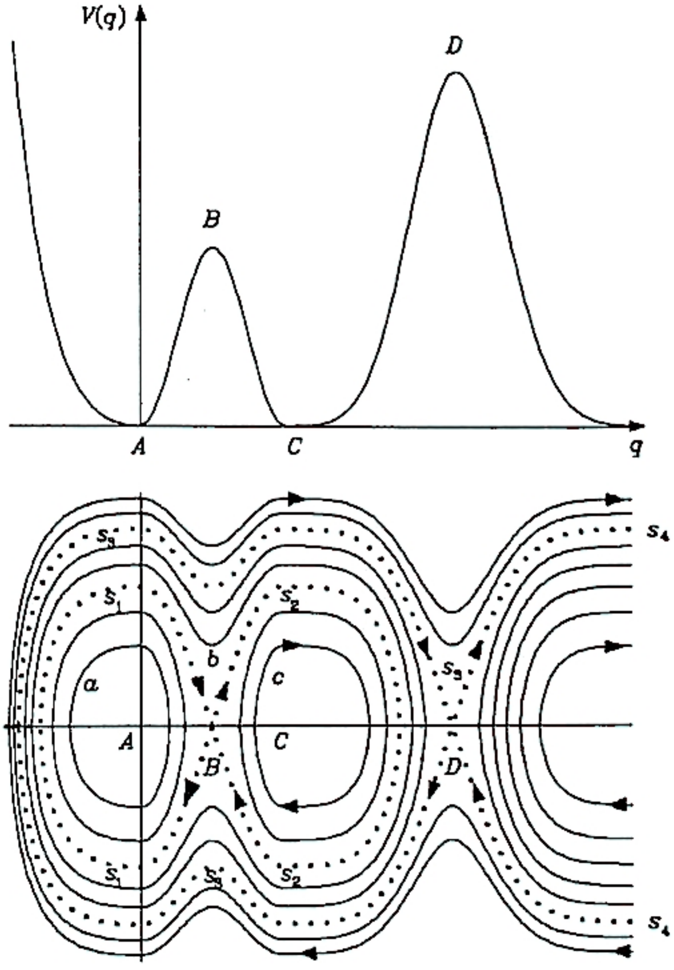


Figure 4.15 The upper diagram is the graph of a potential with two maxima and two minima. The lower diagram shows some contours of the Hamiltonian $H(q,p) = p^2/2 + V(q)$, with the separatrixes depicted by dotted lines.

In this potential there are two minima, A and C, producing the centres in the accompanying phase plane, and there are two maxima, B and D, producing the two saddles.

To sketch the phase portrait, we first need to determine the whole of each of the separatrixes.

Consider the separatrix centred on the maximum at B. The branch s_1, leaving B, represents motion to the left with energy E_B; the particle continues moving to the left until the potential has the value E_B, at which point it is reflected and returns to B, as shown. Similarly, the branch s_2, leaving B, represents motion moving to the right; this is reflected off the higher potential to the right of B, and again returns to B.

Inside the separatrix s_1 are the phase curves corresponding to oscillatory motion in the well centred at A, with energy E such that $E_A < E < E_B$; phase curve a is one example of this motion. Inside the separatrix s_2 are the phase curves corresponding to oscillatory motion in the well centred at C, with energy E such that $E_C < E < E_B$, for example phase curve c.

The separatrix centred on the maximum at D has branches s_3 and s_4. The branch s_3 is similar to s_1 in that it represents motion leaving D, moving to the left, reflecting off the potential at $V(q) = E_D$ and returning to D again. The contours inside s_3 but outside s_1 and s_2, contour b for instance, represent oscillatory motion with energy $E_B < E < E_D$, sufficient to pass over the barrier at B, but insufficient to pass over the barrier at D. The separatrix branch s_4, however, is different: the outward branch represents motion moving away from D to the right, and as the potential to the right of D is always less than E_D, the motion continues to the right and never returns; similarly, the lower branch of s_4 represents motion to the left from infinity. A phase curve 'inside' s_4 represents motion from infinity with energy insufficient to pass over the barrier at D, $E < E_D$, so that it is reflected by the potential at some point $q > D$ and returned to infinity. The phase curve 'outside' s_4 also represents unbounded motion, but with energy $E > E_D$, which is sufficient to pass over the barrier at D, but which is reflected by the potential wall at $q < 0$; notice that as this motion passes over the potential maxima at B and D, the speed passes through successive local minima.

It should be clear from this example yet again that the separatrixes divide phase space into invariant regions in each of which the motion possesses different characteristics. For instance, inside s_3, the motion is bounded and, apart from that on the separatrix through B, periodic, yet outside s_3, the motion is unbounded.

Any potential problem for one degree of freedom can be studied in this way, provided that dV/dq exists and has only simple zeros. On the very rare occasions when it does not, there may be complications.

Exercise 4.29

Describe the qualitative features of the motion of a particle of unit mass moving in the following potentials $V(q)$. In each case, sketch the phase diagram and give the equations of the separatrixes, where they exist.

(i) $V(q) = Aq^2 \exp(-q^2), \quad A > 0.$

(ii) $V(q) = q^{-4} - 2q^{-2}, \quad q > 0.$

4.6 Motion in a circle

As we pointed out in *Unit 1*, the configuration space does not always consist of the whole real line. For rotational motion the configuration space is a circle.

Subsection 1.6.4

It is sometimes convenient to choose a coordinate q which varies in the range $0 \le q \le 2\pi$, with the ends of the range, $q = 0,\ 2\pi$, labelling the same physical point, as shown in Figure 4.16.

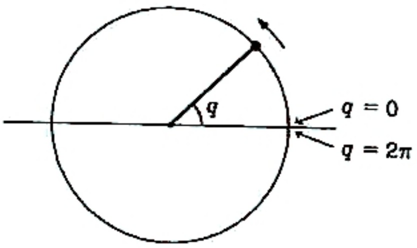


Figure 4.16 Diagram showing a circular configuration space, with the points $q = 0$ and $q = 2\pi$ representing the same physical point.

Usually, however, it is more convenient to allow q to vary over all real values $-\infty < q < \infty$, in which case the points $q + 2n\pi$, $n = 0, \pm 1, \pm 2, \ldots$, all label the same physical point on the circle. This means that all physically observable functions of q must be periodic in q with period 2π; in particular the Hamiltonian, $H(q,p)$, must be periodic:

$$H(q + 2\pi, p) = H(q,p). \tag{4.36}$$

When the generalised coordinate q is an angle, the conjugate momentum has the dimensions of an angular momentum, for the reasons discussed in Exercise 4.8 (page 94). In practice this often means that the conjugate momentum really is an angular momentum.

4.6.1 Free rotations

In this subsection we shall discuss the phase portrait of the system consisting of a body which rotates freely about a fixed axis, under the influence of no external forces of any kind; such a body is often called a *free rotor*. For a free rotor there is no preferred direction, as would be the case for motion under gravity, for example, so the Hamiltonian must be independent of q and must be a function of momentum only: it has the form

Free rotor

Hamiltonian for free rotations

$$H = \frac{p^2}{2\mu}, \tag{4.37}$$

where μ is the moment of inertia, which we discussed in *Unit 1*, and which plays the role that mass plays in rectilinear motion. In this course we shall not be concerned with how the moment of inertia is calculated.

Subsection 1.6.5
We have changed the notation I to μ, because I has another meaning in later units.

The angular momentum p can usually take any value in $(-\infty, \infty)$, positive values of p corresponding to increasing q, and vice versa. The phase space can then be represented as an infinitely long cylinder with p increasing up the cylinder axis, as shown in Figure 4.17.

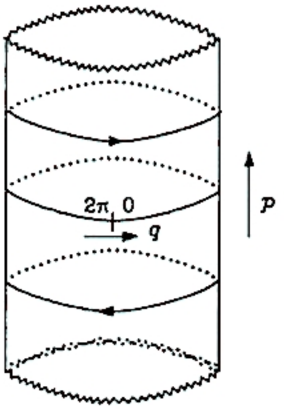


Figure 4.17 Diagram of the phase space cylinder; the momentum coordinate increases along the cylinder axis and the configuration coordinate increases round the cylinder. The arrowheads indicate directions of motion along phase curves for the free rotor.

This representation is not very convenient for graphical representation on two-dimensional paper, so we usually unwrap the cylinder onto the plane, as shown in Figure 4.18.

In this representation, the two sides of the rectangle, $q = \pm\pi$, correspond to the same physical point: a phase curve leaving the rectangle on one side returns on the other side with the same p-coordinate as when it left (see, for instance, Figure 4.20 on page 118). Of course, the cylinder need not be cut at π; any line of constant q will do. Sometimes it is more convenient to make the cut at $q = 2\pi$ and show the range $0 \leq q < 2\pi$. The identification of corresponding points of opposite vertical sides of the rectangle is equivalent to the identification of $q = 0$ and $q = 2\pi$ mentioned above.

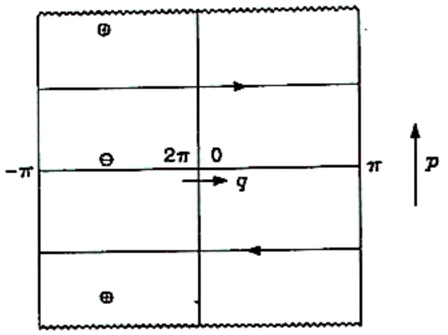


Figure 4.18 Diagram of the phase plane obtained by unwrapping the cylinder of Figure 4.17. In this example the cut is along the line $q = \pm\pi$. The signs ⊕ and ⊖ indicate the 'high' and 'low' ground for the free rotor Hamiltonian, Equation 4.37; also shown are some of its phase curves.

Free rotational motion is particularly simple. Hamilton's equations give

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{\mu}, \qquad \dot{p} = -\frac{\partial H}{\partial q} = 0, \tag{4.38}$$

which can be integrated directly to give

$$p = p_0 \qquad \text{and} \qquad q = \frac{p_0}{\mu}t + q_0, \tag{4.39}$$

where (q_0, p_0) is the state at time $t = 0$. Thus in the cylinder picture, Figure 4.17, the phase curves are just horizontal circles – circles of constant p – and in the plane picture, Figure 4.18, they are straight lines parallel to the q-axis.

There are two interesting points to notice about this solution. First, $p(t)$ is constant because the Hamiltonian is independent of the conjugate variable q, and this is the case because there is no preferred direction. In other words, the system is invariant with respect to changes in q: this leads directly, via Hamilton's equations, to the conjugate variable, p, being constant. This invariance can be described by saying that the transformations $(q, p) \mapsto (q + Q, p)$ are symmetries of the system, for every real number Q. Such symmetries are very important in dynamics, but because they arise most naturally in higher-order systems, we do not discuss them further in this course.

The second important point is that the free rotor has no *isolated* fixed points. Rather, Equations 4.38 show that it has a whole circle of fixed points $p = 0$. Since these are not isolated, they are not simple. As a result, small q-dependent perturbations can completely change the behaviour of the system in the vicinity of this line of fixed points. The example discussed in the next subsection demonstrates this possibility.

Exercise 4.30

(i) Sketch the contours of the Hamiltonian $H = \frac{1}{2}p^2$ in the phase plane of Figure 4.18.

(ii) Find the period of free rotation in terms of the energy E.

4.6.2 The vertical pendulum

It follows from the analysis in *Unit 1* Subsection 1.6.4 that if a time-independent force is applied to the rotor of the previous section, the Hamiltonian of the system assumes the form

$$H(q, p) = \frac{p^2}{2\mu} + V(q), \tag{4.40}$$

where the potential, $V(q)$, is 2π-periodic.

The simplest example of such a system is given by the vertical pendulum in which the mass m at the point P of Figure 4.19 is suspended by a light, stiff rod of length l and is constrained to move in a vertical plane through O in the presence of a uniform downward force F.

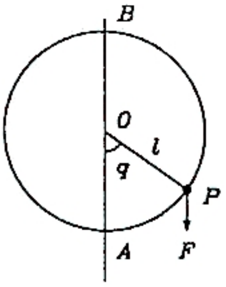


Figure 4.19

If q is the angle between the rod OP and the downward vertical, the potential energy relative to the point A, say, is given by

$$V(q) = Fl(1 - \cos q).$$

The Hamiltonian function is given by the total energy which is, to within an arbitrary additive constant,

$$H(q,p) = \frac{p^2}{2ml^2} - Fl\cos q, \tag{4.41}$$

since $\mu = ml^2$.

This Hamiltonian is of considerable interest in its own right; the pendulum may be somewhat outdated as a timekeeping device, but its motion is easy to visualise and more subtle than may at first appear. But in addition, Hamiltonians of this form are important because they appear in many applications, for example in the theory of molecular rotation, in some approximate theories of planetary motion, in the general theory of systems of many degrees of freedom, and in the important subject of resonantly forced motion, as you will see later. Thus it is important to understand the motion described by this Hamiltonian. We shall now describe the nature of its phase curves, which are shown in Figure 4.20 (overleaf).

Before embarking on a mathematical description of the motion it is very helpful to imagine how the system will behave, in this as in any other case, as this will provide a check on the subsequent analysis. In the present case this is quite easy as we have a familiar concrete realisation of the Hamiltonian system, namely the vertical pendulum. Clearly there are only two fixed points, one in which the rod points vertically downwards, and another in which it points vertically upwards. The first of these is obviously stable, the second even more obviously unstable. Also there are two types of motion: if the mass is moving fast enough it will rotate completely around the circle, whilst if it has insufficient energy to reach the top of the circle, B in Figure 4.19, it will oscillate with $|q| < \pi$. We shall now see how the Hamiltonian given in Equation 4.41 describes such motion.

First, as usual, we find the fixed points. Hamilton's equations are

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{ml^2}, \tag{4.42}$$

$$\dot{p} = -\frac{\partial H}{\partial q} = -Fl\sin q, \tag{4.43}$$

so the fixed points are at

Fixed points of the vertical pendulum

$$p = 0 \qquad \text{and} \qquad q = 0, \pi. \tag{4.44}$$

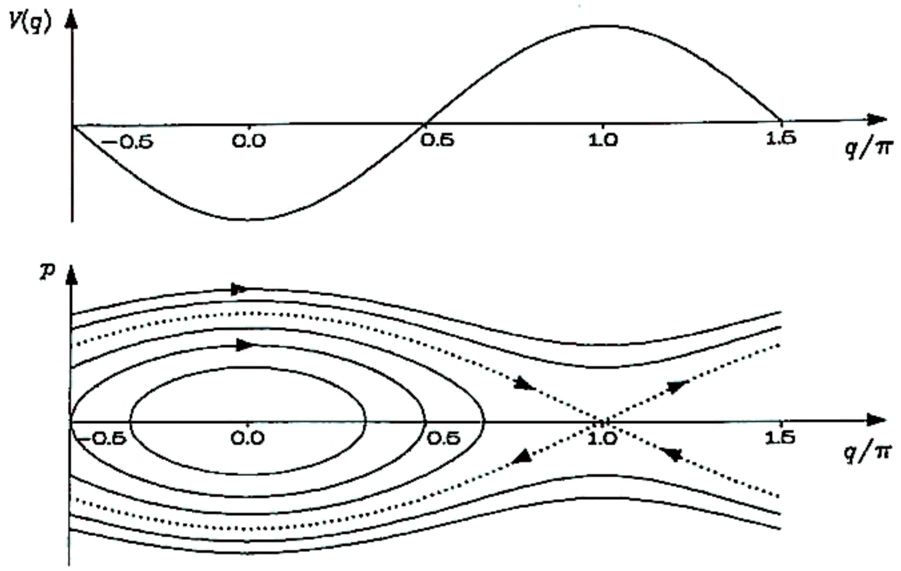


Figure 4.20 The upper diagram is the graph of the potential $V(q) = -Fl\cos q$. The lower diagram shows the contours of the Hamiltonian $H(q,p) = p^2/2ml^2 + V(q)$; this has a centre at $(0,0)$, surrounded by nearly elliptical contours, and a saddle at $(\pi, 0)$, through which passes the separatrix s shown by the dotted lines.

The nature of these fixed points is easily obtained by using the results of Exercise 4.21 (page 106); but a more detailed analysis is both easy and instructive. We expand the Hamiltonian about each fixed point in turn, using the Taylor expansion of $\cos q$, and retain only the quadratic terms. Using the approximation

$$\cos q = 1 - q^2/2 + O(q^4), \tag{4.45}$$

we obtain, apart from irrelevant additive constants,

$$H \simeq \frac{p^2}{2ml^2} + \frac{Fl}{2}q^2, \qquad |q| \ll 1. \tag{4.46}$$

For q near π, put $q = \pi - Q$ to obtain

$$H \simeq \frac{p^2}{2ml^2} - \frac{Fl}{2}Q^2, \qquad Q = \pi - q, \quad |Q| \ll 1. \tag{4.47}$$

On comparing these Hamiltonians with those treated in Examples 4.2 and 4.3 (pages 101 and 103 respectively), it is apparent that the point $(0,0)$ is a centre and is therefore stable, while $(\pi, 0)$ is a saddle and is unstable, as expected.

The phase curves of energy E are given by the solution of $H(q,p) = E$, or

$$p(q,E)^2 = 2ml^2(E + Fl\cos q). \tag{4.48}$$

For energies $E > Fl$, we have $E + Fl\cos q > 0$ for all q, and from this equation we observe that $p(q,E)$ is real and non-zero for every q. Thus for each such energy there are two phase curves defined for all q,

$$p(q,E) = \pm\sqrt{2ml^2(E + Fl\cos q)}, \qquad E > Fl. \tag{4.49}$$

These correspond to the mass P passing completely round the circle, anti-clockwise for $p > 0$ (when $\dot{q} > 0$), and clockwise for $p < 0$ (when $\dot{q} < 0$).

For energies $E < -Fl$, $E + Fl\cos q < 0$ for all q so there are no phase curves. For energies E such that $-Fl \le E < Fl$, there is a range of values of q for which $E + Fl\cos q \ge 0$, defined by the roots of $\cos q = -E/Fl$. For $E \simeq -Fl$, these roots are near $q = 0$, and for these energies the Hamiltonian is given quite well by the expansion in Equation 4.46: the phase curves are nearly elliptical and centred on the origin. With increasing energy the quadratic approximation breaks down, but the phase curves remain closed with turning points at $\cos q = -E/Fl$. Some typical phase curves are shown in Figure 4.20, in which we also show the potential energy, $V(q) = -Fl\cos q$.

As we have shown above, the fixed point at $(0, 0)$, corresponding to point A in Figure 4.19, is a centre. It corresponds to the minimum in the potential at $q = 0$, when the pendulum points vertically downwards; the energy at this fixed point is $-Fl$. The fixed point at $(\pi, 0)$, point B in Figure 4.19, is a saddle and corresponds to the maximum in the potential at $q = \pi$, when the pendulum is vertically upwards; the energy at this fixed point is Fl. The separatrix through this point is the contour of energy $E = E_s = Fl$. Its equation is particularly simple (see Exercise 4.32). It divides phase space into three regions, in each of which the motion is periodic. In the top and bottom regions the pendulum rotates anti-clockwise and clockwise respectively, just as in the case of free rotation, but now the rotation is not uniform. The oscillatory motion about the downward vertical which occurs in the central region is clearly not equivalent to free rotation.

This new motion appears however small the force F might be. The addition of a non-zero force F, no matter how small, has changed the line of fixed points of the Hamiltonian $H = p^2/2\mu$ into a centre and a saddle, and has introduced a distinctly different type of motion – the oscillations about $q = 0$ – not present in the limit $F = 0$. Thus a small perturbation has fundamentally changed the motion in the vicinity of a line of isolated fixed points. This may not seem important, as this example might be considered as rather contrived; nothing could be further from the truth. This simple example pervades much of Hamiltonian dynamics and the changes just observed are of fundamental importance. The reason why the motion of many Hamiltonian systems is so complicated is ultimately related to this simple example. We shall see some of these reasons later, where the 'vertical pendulum' Hamiltonian will arise in a very general context.

Exercise 4.31

Equation 4.46 is equivalent to the 'small oscillations' approximation for a circular pendulum, which replaces the Newtonian equation of motion by an approximating simple harmonic motion equation. What is the period of small oscillations of the system with the Hamiltonian of Equation 4.41 (page 117)?

Exercise 4.32

(i) Show that the equation of the separatrix of the vertical pendulum with Hamiltonian given in Equation 4.41 is

$$p(q) = \pm 2\sqrt{mFl^3}\cos\tfrac{1}{2}q.$$

(ii) Show that the area enclosed by the separatrix is

$$A_s = 16\sqrt{mFl^3}.$$

4.6.3 Rotation and libration

It is clear from the example of the vertical pendulum that periodic motion may be divided into two types, depending upon whether or not the generalised velocity, $\dot{q}$ in the above example, changes sign during the course of the motion.

In general, if the motion of a system is periodic and the generalised coordinate q is expressed as a function of time, then the motion is named

- a *rotation* if $\dot{q}$ always has the same sign, that is, if $\dot{q}$ is never zero;
- a *libration* if $\dot{q}$ changes sign.

Rotation

Libration

Synonyms for libration are ***vibration*** and ***oscillation***.

Vibration, oscillation

Clearly, rotational motion can occur only if the configuration space is periodic; in a non-periodic configuration space, if $\dot{q}$ is never zero then $q(t)$ is either continuously increasing or decreasing, and so cannot produce periodic motion. However, if the configuration space is periodic (for instance, a circle), then either rotational or librational motion can occur. Thus the type of periodic motion possible depends upon the topology of the configuration space.

When the phase space is represented on a plane, as in Figure 4.18 (page 116), motion in any closed curve corresponds to libration. If the configuration space is periodic then rotation appears as a periodic curve with bounded gradient. Figure 4.21 shows examples of both types of periodic motion. The regions of phase space supporting librational and rotational motion are *always* separated by a separatrix.

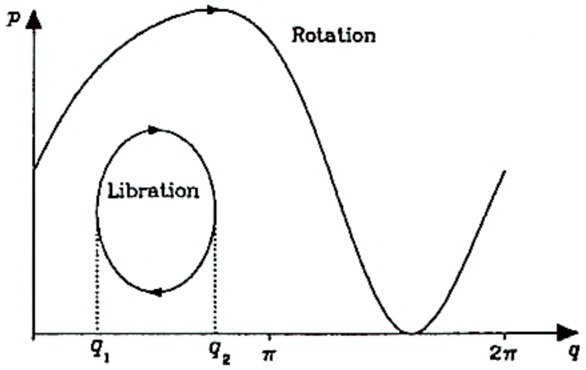


Figure 4.21 Diagram showing a librational and a rotational phase curve for a system with a 2π-periodic configuration space. The rotational phase curve would join up smoothly with itself if the plane were glued into a cylinder.

It is often important to compute the period of a rotation or a libration. This is just the time taken for the phase point to execute one complete cycle of the motion. In the case of a rotation this is the time required for the point's q-coordinate to increase by 2π, whereas for a libration it is the time taken for the point to completely traverse the phase curve once, returning to where it started. For clarity, we shall consider rotational and librational periods separately.

4.6.4 Rotational periods

This is the simpler case, as $\dot{q}$ is never zero. The cases $\dot{q} > 0$ and $\dot{q} < 0$ are not significantly different, but must be treated separately. Suppose that $\dot{q} > 0$, and write $T(E)$ for the period of the rotation when the energy is E. Then $T(E)$ is the time taken for the q-coordinate of the phase point to increase from 0 to 2π. We have

$$T(E) = \int_0^{T(E)} dt = \int_0^{2\pi} \frac{dq}{\dot{q}} = \int_0^{2\pi} \frac{dq}{\partial H/\partial p}, \tag{4.50}$$

where the last integrand is expressed as a function of q using the energy equation $H(q,p) = E$ to obtain the momentum as a single-valued function of q. If the Hamiltonian has the special form

$$H(q,p) = \frac{p^2}{2m} + V(q), \tag{4.51}$$

then the function $p(q,E)$ is easily obtained explicitly,

$$p(q,E) = \pm\sqrt{2m(E - V(q))}, \tag{4.52}$$

so that

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m} = \pm\left\{\frac{2}{m}(E - V(q))\right\}^{1/2}. \tag{4.53}$$

Then in the case $\dot{q} > 0$ we obtain

Rotational period

$$T(E) = \sqrt{\frac{m}{2}} \int_0^{2\pi} \frac{dq}{\sqrt{E - V(q)}}. \tag{4.54}$$

Exercise 4.33

Calculate the period of free rotations by this method, and compare the result with that of Exercise 4.30 (page 116).

Exercise 4.34

There is an unspoken assumption in the argument presented above that the period of rotation is independent of the initial position of the phase point, and so there is no loss of generality in taking the initial value of q to be zero. Show that the time taken for the phase point to go from the point with q-coordinate $q(0)$ to the point with q-coordinate $q(0) + 2\pi$ is

$$\sqrt{\frac{m}{2}} \int_{q(0)}^{q(0)+2\pi} \frac{dq}{\sqrt{E - V(q)}}.$$

By using the fact that $V(q)$ is a periodic function of period 2π, show that the value of this quantity is just $T(E)$, and so confirm that the period is independent of the starting point.

Exercise 4.35

Show that the period of rotational motion of a vertical pendulum can be written in the form of the integral

$$T(E) = \sqrt{\frac{8m}{E + Fl}} \int_0^{\pi/2} \frac{dx}{\sqrt{1 - k^2 \sin^2 x}}, \quad k^2 = \frac{2Fl}{E + Fl}, \quad E > Fl.$$

[Hint: Use the half-angle formula $\cos q = 1 - 2\sin^2 \frac{1}{2}q$.]

Exercise 4.36

Show that if $\dot{q} < 0$, then the period can be written in the form

$$T(E) = \int_0^{-2\pi} \frac{dq}{\dot{q}} = -\sqrt{\frac{m}{2}} \int_0^{-2\pi} \frac{dq}{\sqrt{E - V(q)}},$$

and that this reduces to Equation 4.54 by the change of variable $q' = 2\pi + q$.

4.6.5 Librational periods

It is slightly more difficult to determine a librational period as the change in sign of $\dot{q}$ needs to be explicitly accounted for. Suppose the phase curves are as shown in Figure 4.21, with the motion bounded between $q_1(E) \le q \le q_2(E)$, where $q_1(E)$ and $q_2(E)$ are roots of the equation $V(q) = E$. Then for each value of q there are two values of p for points on the curve. We say that the curve has two branches, one corresponding to the larger, the other to the smaller value of p. On the upper branch $\dot{q} = \partial H/\partial p > 0$, and on the lower branch $\dot{q} = \partial H/\partial p < 0$. If we now assume the Hamiltonian to have the special form of Equation 4.51 then the expression for on each branch is given in Equation 4.53. The time to travel between q_1 and q_2 on the upper branch is

$$T_1(E) = \int_{q_1}^{q_2} \frac{dq}{\dot{q}} = \sqrt{\frac{m}{2}} \int_{q_1}^{q_2} \frac{dq}{\sqrt{E - V(q)}}, \tag{4.55}$$

and the time taken to travel between q_2 and q_1 on the lower branch is

$$T_2(E) = \int_{q_2}^{q_1} \frac{dq}{\dot{q}} = -\sqrt{\frac{m}{2}} \int_{q_2}^{q_1} \frac{dq}{\sqrt{E - V(q)}} = T_1(E). \tag{4.56}$$

The period is therefore

$$\begin{aligned} T(E) &= T_1(E) + T_2(E) \\ &= \sqrt{2m} \int_{q_1(E)}^{q_2(E)} \frac{dq}{\sqrt{E - V(q)}}. \end{aligned} \tag{4.57}$$

Librational period

In the case of a system of mechanical type, treated here, the Hamiltonian is an even function of p, so the momenta on the upper and lower branches are equal in magnitude but of opposite sign, and $T_1 = T_2$. For a more general Hamiltonian this need not be the case.

Exercise 4.37

Compute the period of libration (oscillation) of a linear oscillator (see Example 4.2, page 101) by this method.

Generally, the period integrals 4.54 and 4.57 cannot be evaluated in terms of known functions, so suitable numerical methods are described in the glossary section of the Handbook under 'periods'.

4.7 Area-preserving flows

We introduced Hamiltonian systems in Section 4.1 as a generalisation of Newtonian systems, but they can be considered from quite a different point of view, as those systems whose evolution, or flow, preserves area in phase space. This is a very important property of second-order Hamiltonian systems and will be used extensively in subsequent units. In this section we show that systems which satisfy Hamilton's equations preserve phase space area and, because this result is so important, we shall demonstrate it twice, once geometrically for conservative systems and once analytically for systems whose Hamiltonians may be explicit functions of time.

The area-preserving property may be stated explicitly as follows.

> Consider any second-order Hamiltonian system, not necessarily conservative. For any bounded region R in phase space, let $\mathcal{A}(R)$ denote the area of R. Let $R(t)$ be the region of phase space obtained by taking a region $R(t_0)$ at time t_0 and following its points as the system evolves through time t: thus $R(t)$ consists of the points $(q(t), p(t))$ such that $(q(t_0), p(t_0)) \in R(t_0)$. Then
>
> $$\mathcal{A}(R(t)) = \mathcal{A}(R(t_0)).$$

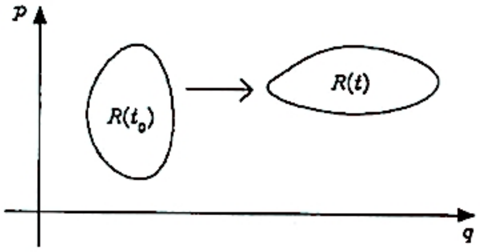


Figure 4.22

It may be objected that, since phase space is an abstract space, it is not clear what is meant by the area of a region of it. We simply mean the area calculated as though phase space were an ordinary two-dimensional Euclidean space with Cartesian coordinates q and p; thus

$$\mathcal{A}(R) = \iint_R dq\, dp.$$

This applies even in the case of rotational motion, provided the phase space is taken to be not a cylinder, but the plane onto which the cylinder can be unwrapped, as described in Section 4.6.

In the conservative case we can state this property in a more direct form, using the concept of the flow of an autonomous system defined in *Unit 3* Section 3.2.

> **Preservation of area by the flow of a conservative system**
>
> Let ϕ_t be the flow of a conservative second-order Hamiltonian system. Then for any region R of phase space and for any time t,
>
> $$\mathcal{A}(\phi_t(R)) = \mathcal{A}(R).$$

4.7.1 Preservation of area by conservative systems

To prove that flows of conservative systems preserve area, we take advantage of the fact that the phase curves are the contours of the Hamiltonian function. The first step is to show that the area of a suitably oriented area element is preserved under the Hamiltonian flow.

Consider an element of area of phase space, R, in the shape of a parallelogram, lying between two neighbouring contours, of energy E and $E + \delta E$, as shown in Figure 4.23.

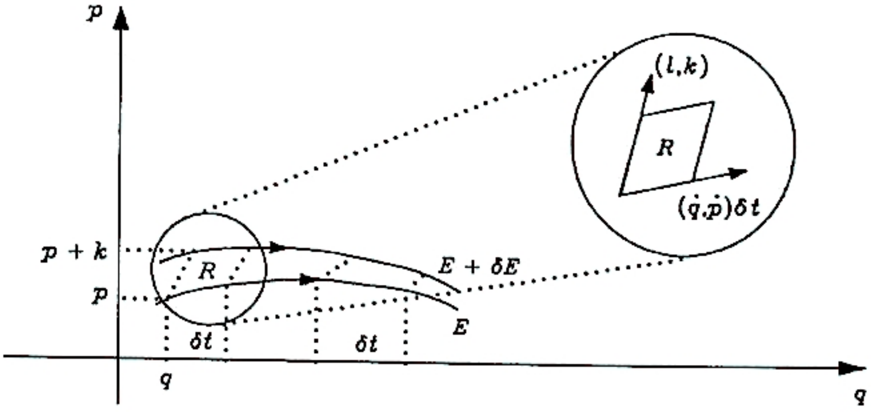


Figure 4.23

Suppose that the lower-left vertex of R has coordinates (q,p). The lower-right vertex lies on the same phase curve as (q,p). Its coordinates are therefore approximately $(q+\delta q, p+\delta p)$, where

$$\delta q = \dot{q}\,\delta t = \frac{\partial H}{\partial p}(q,p)\,\delta t, \qquad \delta p = \dot{p}\,\delta t = -\frac{\partial H}{\partial q}(q,p)\,\delta t \tag{4.58}$$

for some δt which is the time separation between the two vertices: a point initially at the lower-left vertex of R, moving under the flow of the system, would arrive at the lower-right vertex after time δt had elapsed.

The upper-left vertex lies on the contour of energy $E + \delta E$. Suppose that its coordinates are $(q+l, p+k)$, where l and k are $O(\delta q)$; then

$$\begin{aligned} H(q+l,p+k) &= E + \delta E \\ &= H(q,p) + l\frac{\partial H}{\partial q}(q,p) + k\frac{\partial H}{\partial p}(q,p) + \text{higher-order terms} \\ &= E + l\frac{\partial H}{\partial q}(q,p) + k\frac{\partial H}{\partial p}(q,p) + \text{higher-order terms.} \end{aligned} \tag{4.59}$$

Thus, on ignoring the higher-order terms,

$$l\frac{\partial H}{\partial q}(q,p) + k\frac{\partial H}{\partial p}(q,p) = \delta E, \tag{4.60}$$

approximately. As was shown in *Unit 1*, the area of this parallelogram is the determinant of the matrix whose columns are the components of vectors along two of its adjacent sides; so

Subsection 1.4.3

$$\begin{aligned} \mathcal{A}(R) &= \begin{vmatrix} \dot{q}\,\delta t & l \\ \dot{p}\,\delta t & k \end{vmatrix} \\ &= (\dot{q}k - \dot{p}l)\delta t \\ &= \left(k\frac{\partial H}{\partial p}(q,p) + l\frac{\partial H}{\partial q}(q,p)\right)\delta t \\ &= \delta E\,\delta t. \end{aligned} \tag{4.61}$$

Under the flow, the vertices of the parallelogram move to points which remain on the same contours; thus δE, the difference between the energies of the two contours, does not change. Moreover, because the system is autonomous, the time separation between points on the same contour does not change, and so δt does not change. Thus the area of the elementary parallelogram is preserved under the flow.

As any region may be broken down into such elementary parallelograms, we see that the flow preserves area. This completes the proof in the conservative case.

Exercise 4.38

Example 4.2, page 101

Consider the linear oscillator. Show that the solution to Hamilton's equations with initial conditions $q(0) = Q$, $p(0) = P$, is

$$q(t) = Q\cos\omega t + \frac{P}{m\omega}\sin\omega t, \qquad p(t) = -m\omega Q\sin\omega t + P\cos\omega t.$$

Observe that, for fixed t, this defines a mapping $(Q, P) \mapsto (q(t), p(t))$, which is linear. Show that the matrix corresponding to this linear mapping has unit determinant, and interpret this result.

Exercise 4.39

The result of this subsection may be interpreted in terms of a change of coordinates in phase space, as follows. The new, curvilinear, coordinates will be the energy and the time; they will be denoted by E and t as usual, though it must be remembered that these variables are now being used as coordinates in phase space. The t-axis can be any curve, τ, which cuts the contours of H transversely (that is, which is nowhere tangent to a contour of H). Then, given any point P in phase space close enough to τ, the contour on which P lies will intersect τ in a single point, say P_0. The new coordinates (t, E) of P are the energy E of the contour on which it lies, and the time separation t of P from P_0. Show that

(i) this change of coordinates is area-preserving (this is essentially the content of the argument above);

(ii) in terms of the new coordinates, the flow of the system through time T is simply the time-translation $(t, E) \mapsto (t + T, E)$.

Deduce that the flow of a conservative system is area-preserving.

4.7.2 Preservation of area by time-dependent systems

A system influenced by time-dependent forces, or which is represented in a rotating or other non-inertial reference frame, as discussed in the next unit, has a Hamiltonian function $H(q, p, t)$ which depends explicitly upon the time. For a particular motion of such a system, the rate of change in the value of the Hamiltonian with time is given by

Compare Exercise 4.13, page 95.

$$\frac{dH}{dt} = \frac{\partial H}{\partial q}\dot{q} + \frac{\partial H}{\partial p}\dot{p} + \frac{\partial H}{\partial t}, \tag{4.62}$$

and on using Hamilton's equations of motion this becomes

$$\frac{dH}{dt} = \frac{\partial H}{\partial q}\frac{\partial H}{\partial p} + \frac{\partial H}{\partial p}\left(-\frac{\partial H}{\partial q}\right) + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t}. \tag{4.63}$$

That is, the time variation of the value of H in the motion is equal to the time variation of H at the current point in phase space, *considered as fixed*. The value of the Hamiltonian is not conserved; the contours of H depend upon time and are now not related to the motion; in particular, the argument of the previous subsection cannot be used to demonstrate preservation of area. However, area *is* preserved, as we shall now show.

The following argument is valid for any two-dimensional flow, and because the general theory emphasises the reason why Hamiltonian flows are area-preserving, we first consider a general second-order system with equations of motion

$$\dot{x} = v_x(x, y, t), \qquad \dot{y} = v_y(x, y, t),$$

which are not necessarily Hamiltonian. We shall follow the time development of a region of phase space, of small area, under the flow of the system; call this region, at time t, $R(t)$. Initially, at time $t = t_0$, the region is $R(t_0)$. Of particular interest is its progress from the time t to a slightly later time $t + \delta t$; so we need the relation between the areas of the regions $R(t)$ and $R(t + \delta t)$ and will eventually let δt tend to zero.

Exercise 4.38 gives an explicit example of this process, with $t_0 = 0$; however, the system considered there is autonomous, and the map is linear, neither of which properties are assumed in the present discussion.

The point of interest is how the area of the region changes with time. The area of the region at time $t + \delta t$ is

$$\mathcal{A}(t + \delta t) = \iint_{R(t+\delta t)} dx\, dy.$$

We can consider the flow of the system through time δt as a transformation from the region $R(t)$ to the region $R(t + \delta t)$, sending a point (X, Y) in $R(t)$ to the point (x, y) in $R(t + \delta t)$. In other words, (X, Y) can be considered as the initial condition on a trajectory $(x(X, Y, t + \delta t), y(X, Y, t + \delta t))$, where

$$x(X, Y, t) = X \qquad \text{and} \qquad y(X, Y, t) = Y;$$

and the transformation from $R(t)$ to $R(t + \delta t)$ is given by

$$(X, Y) \longmapsto (x(X, Y, t + \delta t), y(X, Y, t + \delta t)).$$

Then by the general theory of changes of area described in *Unit 1*, the area of $R(t + \delta t)$ may be expressed in terms of the Jacobian of the transformation $(X, Y) \longmapsto (x, y)$ as follows:

Subsection 1.4.6

$$\mathcal{A}(t + \delta t) = \iint_{R(t+\delta t)} dx\, dy = \iint_{R(t)} dX\, dY \frac{\partial(x, y)}{\partial(X, Y)}. \tag{4.64}$$

Since δt is small we can expand the functions $x(X, Y, t + \delta t)$ and $y(X, Y, t + \delta t)$ using Taylor's theorem:

$$\begin{aligned} x(X, Y, t + \delta t) &= X + \dot{x}(X, Y, t)\delta t + O(\delta t^2) \\ &= X + v_x(X, Y, t)\delta t + O(\delta t^2), \\ y(X, Y, t + \delta t) &= Y + \dot{y}(X, Y, t)\delta t + O(\delta t^2) \\ &= Y + v_y(X, Y, t)\delta t + O(\delta t^2). \end{aligned}$$

So the Jacobian determinant in Equation 4.64 can be written in the form

$$\begin{aligned} \frac{\partial(x, y)}{\partial(X, Y)} &= \begin{vmatrix} 1 + \delta t \dfrac{\partial v_x}{\partial X} & \delta t \dfrac{\partial v_x}{\partial Y} \\ \delta t \dfrac{\partial v_y}{\partial X} & 1 + \delta t \dfrac{\partial v_y}{\partial Y} \end{vmatrix} \\ &= 1 + \delta t \left(\frac{\partial v_x}{\partial X} + \frac{\partial v_y}{\partial Y} \right) + O(\delta t^2). \end{aligned} \tag{4.65}$$

Eventually we want to let both δt and the area tend to zero. There is no problem in letting $\delta t \to 0$; but there are formal difficulties in letting $\mathcal{A}(t) \to 0$, which can be circumvented by the simple device of considering, not the area, but the ratio $a(t)$ of the area of the region at time t to its area at the initial time t_0:

$$a(t) = \frac{\mathcal{A}(t)}{\mathcal{A}(t_0)}. \tag{4.66}$$

Then $a(t)$ tends to a finite limit as $\mathcal{A}(t_0) \to 0$.

Thus on using Equation 4.65 in Equation 4.64, and dividing by $\mathcal{A}(t_0)$, we obtain the following expression relating $a(t+\delta t)$ to $a(t)$:

$$\begin{aligned}
a(t+\delta t) &= \frac{1}{\mathcal{A}(t_0)} \iint_{R(t)} dX\, dY \left(1 + \delta t \left(\frac{\partial v_x}{\partial X} + \frac{\partial v_y}{\partial Y}\right)\right) + O(\delta t^2) \\
&= a(t) + \frac{\delta t}{\mathcal{A}(t_0)} \iint_{R(t)} dX\, dY \left(\frac{\partial v_x}{\partial X} + \frac{\partial v_y}{\partial Y}\right) + O(\delta t^2).
\end{aligned}$$

Now make the region $R(t_0)$, and hence $R(t)$, sufficiently small that the integrand varies little over the region; then it may be taken outside the integral and the limit $\delta t \to 0$ taken to give

$$\frac{da}{dt} = a(t)\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}\right). \tag{4.67}$$

This is the required general expression for the rate of change of the ratio of a small area of phase space.

Specialising now to Hamiltonian systems, we have

$$\frac{\partial v_x}{\partial x} = -\frac{\partial v_y}{\partial y}$$

by the mixed derivative test, Equation 4.15 (page 93). Thus for any Hamiltonian system,

$$\frac{da}{dt} = 0;$$

$a(t)$ is constant, $\mathcal{A}(t) = \mathcal{A}(t_0)$, and so the flow is area-preserving.

This argument shows that the flow of *any* Hamiltonian system is area-preserving. Conversely, if the flow of a given second-order system is area-preserving then, by Equation 4.67, its velocity field satisfies the mixed derivative test and so the system is Hamiltonian. Thus being Hamiltonian, and having an area-preserving flow, are equivalent properties of a second-order system.

This ends the similarity between time-independent and time-dependent second-order Hamiltonian systems. The general theory of the latter is incomplete, and only for special classes of time-dependence does a general theory exist. We shall deal with some of these systems later in the course.

In this section we have stressed the importance of the area-preservation property; you will see in future units that this property is essential to the development of the theory of Hamiltonian systems.

4.8 Higher-order Hamiltonian systems

Most dynamical systems have phase spaces of dimension greater than two, so you may be wondering whether there is a natural generalisation of Hamiltonian theory to systems of order greater than two, and if so what is the generalisation of the area-preservation property. In the last few paragraphs of this unit we try to provide a brief answer to this question. We include this material for the sake of completeness only; it is not required for the remainder of the course, nor is it examinable.

It is possible to generalise Hamilton's equations, and to define a Hamiltonian system, only when the phase space has even dimension. In such a case, where the dimension is $2N$ say, the $2N$ coordinates are conventionally written

$$(q_1, q_2, \ldots, q_N, p_1, p_2, \ldots, p_N).$$

The qs and ps are called conjugate variables, and the system is said to have N degrees of freedom. For any function $H(q_1, q_2, \ldots, q_N, p_1, p_2, \ldots, p_N, t)$ Hamilton's equations of motion are

$$\frac{dq_k}{dt} = \frac{\partial H}{\partial p_k}, \qquad \frac{dp_k}{dt} = -\frac{\partial H}{\partial q_k}, \qquad k = 1, 2, \ldots, N.$$

A system is said to be Hamiltonian if its equations of motion can be put in Hamiltonian form by the choice of a suitable Hamiltonian function (and an appropriate division of coordinates into conjugate sets). Any Newtonian mechanical system in which the forces may be derived from a potential is a Hamiltonian system, with the total energy as Hamiltonian function, and the configuration coordinates and the corresponding components of momentum as conjugate variables. The motion of a single particle in three-dimensional space is an obvious example; but in fact the motion of several interacting particles may be treated in the same way, provided only that the forces of mutual interaction, as well as any external forces, are derivable from a potential. Since the gravitational interaction between any two particles has this property, this means that the whole solar system may be modelled as a Hamiltonian system. There are many other important applications of Hamiltonian theory of higher order.

The identification of the phase curves with the contours of conservative Hamiltonians, which has played such an important role in this unit, works because we are dealing with second-order systems. For higher-order Hamiltonian systems, the Hamiltonian is also constant along phase curves. However, the equation $H =$ constant then defines a surface, not a curve, and we can deduce only that the phase curves lie on surfaces of constant H.

The question that arises from the discussion of the previous section is how the area-preservation property generalises to higher-order Hamiltonian systems. The most obvious generalisation of area when going from a two- to a many-dimensional phase space is a volume; and there is a theorem, known as Liouville's theorem, which states that for a Hamiltonian system with N degrees of freedom, the volume of a bounded region R is constant if the region is allowed to move with the flow. Unfortunately, the converse is not true: not all volume-preserving flows are Hamiltonian – this is obvious because this property is not dependent upon the phase space dimension being even. Thus Liouville's theorem is not the correct generalisation of the area-preservation property, although it is an important property of Hamiltonian systems and has significant consequences in applications, for example in thermodynamics.

The correct generalisation is more subtle. Consider a bounded region R in the $2N$-dimensional phase space, and for each k, $k = 1, 2, \ldots, N$, the projection of R onto the coordinate plane defined by the pair of conjugate coordinates (q_k, p_k). In each such coordinate plane the region R will be projected into a bounded region with a closed curve enclosing a two-dimensional region, say R_k. Then, when the system is Hamiltonian, the *sum* of these projected areas remains constant as the region R evolves in time. This result may be expressed as follows:

$$\sum_{k=1}^{N} \iint_{R_k} dq_k \, dp_k = \text{constant}. \qquad (4.68)$$

Conversely, a system which has this property (for every region R) is necessarily Hamiltonian.

You can see that for systems with one degree of freedom, Liouville's theorem and Equation 4.68 are equivalent. But for systems of two or more degrees of freedom, the generalisation of the theory for one degree of freedom systems, begun in the present unit and to be developed in Block II, must be based on Equation 4.68 rather than Liouville's theorem.

Solutions to the Exercises in Unit 4

Solution 4.1

(i) Hamilton's equations give

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m}, \qquad \dot{p} = -\frac{\partial H}{\partial q} = 0.$$

These are therefore the Hamiltonian equations of motion for a free particle, Equations 4.1 and 4.2 (page 90), with $F = 0$.

(ii) Hamilton's equations give

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m}, \qquad \dot{p} = -\frac{\partial H}{\partial q} = -k,$$

which, by comparison with Equations 4.1 and 4.2, are the equations of motion for a particle acted on by a constant force $F = -k$.

Solution 4.2

Consider two potentials $V_1(q)$ and $V_2(q)$ which differ by a constant. Their Hamiltonians are

$$H_1 = \frac{p^2}{2m} + V_1(q) \qquad \text{and} \qquad H_2 = \frac{p^2}{2m} + V_2(q).$$

Since the potentials differ only by a constant, so do the Hamiltonians; Hamilton's equations involve only derivatives of the Hamiltonian, so that Hamilton's equations for these two Hamiltonians are the same.

Solution 4.3

The phase velocity is given by

$$\mathbf{v} = \left(\frac{\partial H}{\partial p}, -\frac{\partial H}{\partial q}\right).$$

The gradient, on the other hand, is

$$\operatorname{grad} H = \left(\frac{\partial H}{\partial q}, \frac{\partial H}{\partial p}\right).$$

It is easy to see that the former is obtained from the latter by a rotation through a right-angle; for example, $\mathbf{v}$ is clearly orthogonal to $\operatorname{grad} H$ and of the same length. Perhaps the best way of answering this question is to note that $\mathbf{v}$ can be obtained from $\operatorname{grad} H$ by the application of the matrix

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

(where the two vectors must be treated as column vectors), and this is the matrix of a *clockwise* rotation through a right-angle.

Solution 4.4

Hamilton's equations are

$$\dot{q} = \frac{p}{m}, \qquad \dot{p} = -kq + A\cos\Omega t.$$

Differentiation of the first equation and substitution then gives

$$m\ddot{q} + kq = A\cos\Omega t.$$

Solution 4.5

(i) $\dot{q} = b, \quad \dot{p} = -a.$

(ii) $\dot{q} = 2bq + 2cp, \quad \dot{p} = -2aq - 2bp.$

(iii) $\dot{q} = \cos q, \quad \dot{p} = p\sin q.$

Solution 4.6

Application of the test in the previous exercise gives the following.

(i) $\dfrac{\partial v_x}{\partial x} = 0, \quad \dfrac{\partial v_y}{\partial y} = 0;$ Hamiltonian, $H = \frac{1}{2}(-x^2 + y^2).$

(ii) $\dfrac{\partial v_x}{\partial x} = 1, \quad \dfrac{\partial v_y}{\partial y} = 1;$ not Hamiltonian.

(iii) $\dfrac{\partial v_x}{\partial x} = 0, \quad \dfrac{\partial v_y}{\partial y} = 0;$ Hamiltonian, $H = \frac{1}{2}(x^2 + y^2).$

(iv) $\dfrac{\partial v_x}{\partial x} = 0, \quad \dfrac{\partial v_y}{\partial y} = 0;$ Hamiltonian, $H = \displaystyle\int dy\, g(y) - \int dx\, f(x).$

(v) $\dfrac{\partial v_x}{\partial x} = 0, \quad \dfrac{\partial v_y}{\partial y} = 0;$ Hamiltonian, $H = \frac{1}{2}(x^2 t^2 + y^2 t).$

(vi) $\dfrac{\partial v_x}{\partial x} = 1 - y, \quad \dfrac{\partial v_y}{\partial y} = -1 + x;$ not Hamiltonian.

Solution 4.7

Since $H_1 - H_2 = f(t)$ we have

$$\frac{\partial H_1}{\partial q} = \frac{\partial H_2}{\partial q} \quad \text{and} \quad \frac{\partial H_1}{\partial p} = \frac{\partial H_2}{\partial p},$$

so the equations of motion of H_1 and H_2 are identical.

Solution 4.8

From either of Hamilton's equations of motion,

$$\dot{q} = \frac{\partial H}{\partial p} \quad \text{or} \quad \dot{p} = -\frac{\partial H}{\partial q},$$

we have

$$\frac{\mathrm{Dim}(q)}{\mathrm{Dim}(t)} = \frac{\mathrm{Dim}(H)}{\mathrm{Dim}(p)} \quad \text{or} \quad \mathrm{Dim}(qp) = \mathrm{Dim}(Ht),$$

so that qp has the dimensions of *energy* × *time*. Furthermore,

$$\mathrm{Dim}(Ht) = [\mathrm{ML^2T^{-1}}],$$

which are the dimensions of angular momentum (length × linear momentum). If q an angle, it is dimensionless, so $\mathrm{Dim}(qp) = \mathrm{Dim}(p) = [\mathrm{ML^2T^{-1}}]$.

Solution 4.9

On differentiating the first equation we obtain

$$\ddot{x} = \dot{y} = -x, \quad \text{or} \quad \ddot{x} + x = 0,$$

which has general solution

$$x(t) = A\cos(t+\alpha), \qquad \dot{x}(t) = -A\sin(t+\alpha),$$

for some constants A and α. The Hamiltonian is $H(x,y) = (x^2+y^2)/2$, and on a solution, it has the value

$$H(x,y) = \tfrac{1}{2}\left\{A^2\cos^2(t+\alpha) + A^2\sin^2(t+\alpha)\right\} = \tfrac{1}{2}A^2,$$

which is clearly constant.

Solution 4.10

Differentiate the first equation to give $\ddot{q} = \dot{p}$; then

$$\ddot{q} = \dot{p} = -(\omega^2 q + 2\alpha\omega\dot{q}), \qquad \text{or} \qquad \ddot{q} + 2\alpha\omega\dot{q} + \omega^2 q = 0.$$

Suppose that a Hamiltonian $H(q,p)$ exists; then

$$\dot{q} = p = \frac{\partial H}{\partial p}, \qquad \dot{p} = -(\omega^2 p + 2\alpha\omega p) = -\frac{\partial H}{\partial q}.$$

From the first of these equations $\partial^2 H/\partial p \partial q = 0$ and from the second $\partial^2 H/\partial q \partial p = 2\alpha\omega$. Thus the system is Hamiltonian only if $\alpha = 0$, that is, if there is no damping.

Solution 4.11

Treating E as a function of (q,p) and using the chain rule we obtain

$$\begin{aligned}\frac{dE}{dt} &= \frac{\partial E}{\partial q}\dot{q} + \frac{\partial E}{\partial p}\dot{p} \\ &= \omega^2 qp - p(\omega^2 q + 2\alpha\omega p) = -2\alpha\omega p^2.\end{aligned}$$

Solution 4.12

Hamilton's equations are

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m}e^{-2\alpha\omega t}, \qquad \dot{p} = -\frac{\partial H}{\partial q} = -m\omega^2 q e^{2\alpha\omega t}.$$

On differentiating the first equation, and multiplying through by m, we obtain

$$\begin{aligned}m\ddot{q} &= (\dot{p} - 2\alpha\omega p)e^{-2\alpha\omega t} \\ &= -m\omega^2 q - 2\alpha\omega m\dot{q},\end{aligned}$$

where we have used Hamilton's equations to substitute for $\dot{p}$ and p to obtain the last equation. This is the equation of motion for a damped linear oscillator, which can therefore be described by a time-dependent Hamiltonian.

Solution 4.13

Differentiating the Hamiltonian $H(q,p,t)$ with respect to time gives

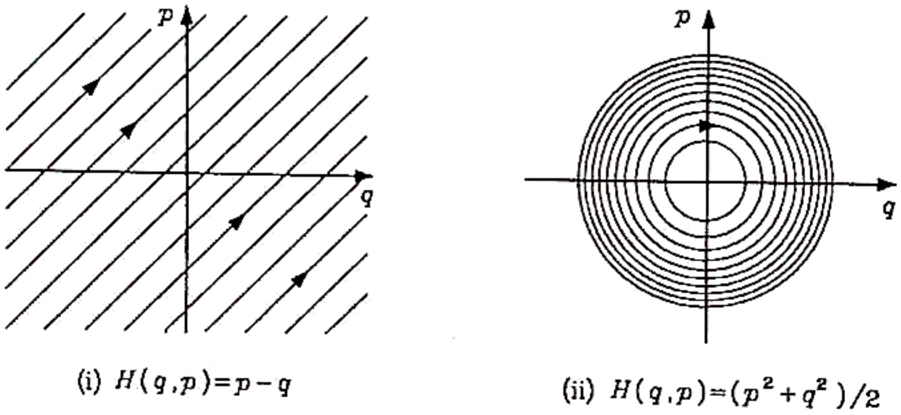
$$\frac{dH}{dt} = \frac{\partial H}{\partial q}\dot{q} + \frac{\partial H}{\partial p}\dot{p} + \frac{\partial H}{\partial t};$$

but Hamilton's equations are $\dot{q} = \partial H/\partial p$ and $\dot{p} = -\partial H/\partial q$, so

$$\frac{dH}{dt} = \left(\frac{\partial H}{\partial q}\right)\left(\frac{\partial H}{\partial p}\right) + \left(\frac{\partial H}{\partial p}\right)\left(-\frac{\partial H}{\partial q}\right) + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t}.$$

If the Hamiltonian is conserved along every motion, then $\partial H/\partial t = 0$, and the system is autonomous.

Solution 4.14



(i) $H(q,p) = p - q$

(ii) $H(q,p) = (p^2 + q^2)/2$

Figure S4.1

Solution 4.15

Since the point (q, p) lies on the contour of energy E, $H(q,p) = E$. Take a nearby point $(q + \delta q, p + \delta p)$ on the contour of energy $E + \delta E$, so that $H(q + \delta q, p + \delta p) = E + \delta E$. Expand this last equation, ignore second-order terms, and use Hamilton's equations, to obtain

$$\delta E = \delta q \frac{\partial H}{\partial q} + \delta p \frac{\partial H}{\partial p} = -\dot{p}\,\delta q + \dot{q}\,\delta p. \qquad \text{(S4.1)}$$

The distance, d, between these two points is given by

$$d^2 = \delta q^2 + \delta p^2,$$

and we need to find the minimum value of this expression, subject to the relation in Equation S4.1. Notice that

$$\begin{aligned} \delta E^2 &= (-\dot{p}\,\delta q + \dot{q}\,\delta p)^2 \\ &= \dot{p}^2 \delta q^2 + \dot{q}^2 \delta p^2 - 2\dot{p}\dot{q}\,\delta p\,\delta q \\ &= (\dot{q}^2 + \dot{p}^2)(\delta q^2 + \delta p^2) - (\dot{q}\,\delta q + \dot{p}\,\delta p)^2 \\ &= |\mathbf{v}|^2 d^2 - (\dot{q}\,\delta q + \dot{p}\,\delta p)^2. \end{aligned}$$

Thus

$$d^2 = \frac{\delta E^2 + (\dot{q}\,\delta q + \dot{p}\,\delta p)^2}{|\mathbf{v}|^2}.$$

The minimum value of d is obtained when the last bracket is zero, giving

$$\Delta = \frac{\delta E}{|\mathbf{v}|}.$$

Notice that the minimum value is obtained when $(\delta q, \delta p)$ is orthogonal to the phase velocity, that is, when it lies in the direction of the gradient of H; this result should not be surprising in view of the fact that grad H is in the direction in which H is changing fastest.

Solution 4.16

(i) If $H(q,p) = E$ we can consider p as a function of q and E. Partial differentiation with respect to E then gives (since q and E are to be treated as independent variables)

$$\frac{\partial H}{\partial p}\frac{\partial p}{\partial E} = 1.$$

But as $\dot{q} = \partial H/\partial p$, the result follows.

(ii) Partial differentiation of $H(q,p) = E$ with respect to q gives

$$\frac{\partial H}{\partial q} + \frac{\partial H}{\partial p}\frac{\partial p}{\partial q} = 0,$$

and on using Hamilton's equations, the result follows.

(At a point where $\dot{q} = 0$, $\dot{p}$ is finite (because $\dot{p} = -\partial H/\partial q$); consequently, by part (ii), $|\partial p/\partial q|$ is infinite, so that the phase curve at such a point is parallel to the p-axis, as one might expect. This remark is relevant to the discussion of turning points in the following subsection.)

Solution 4.17

The function $q(t)$ is given by

$$q(t) = A_+ e^{\gamma t} + A_- e^{-\gamma t}, \qquad \gamma > 0,$$

so for t large and positive $q(t) \simeq A_+ e^{\gamma t}$, hence on curves A and B, $A_+ > 0$ and on curves C and D, $A_+ < 0$. For t large and negative $q(t) \simeq A_- e^{-\gamma t}$, so A_- is negative on A and D and positive on C and B.

In summary,

Curve	A	B	C	D
sign of A_+	$+$	$+$	$-$	$-$
sign of A_-	$-$	$+$	$+$	$-$
sign of energy E	$+$	$-$	$+$	$-$

or, diagrammatically,

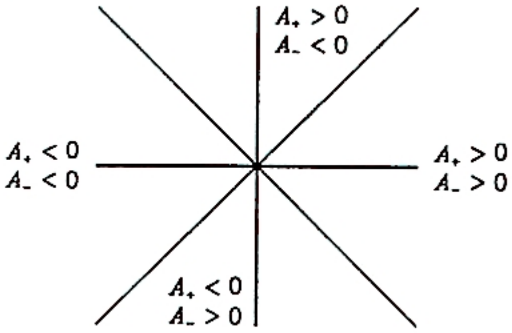


Figure S4.2

Solution 4.18

On the separatrix $E = 0$, so that either $A_+ = 0$ or $A_- = 0$. If $A_+ = 0$, $q(t) = A_- e^{-\gamma t}$ and $p(t) = -m\gamma A_- e^{-\gamma t}$, or $p = -m\gamma q$ and the motion is towards the origin; notice that the fixed point is not reached in a finite time. If $A_- = 0$ then $q(t) = A_+ e^{\gamma t}$ and $p(t) = m\gamma A_+ e^{\gamma t}$, or $p = m\gamma q$ and the motion is away from the origin.

Solution 4.19

It is simple to check that the given curve is a solution to Hamilton's equations:

$$\dot{q} = \gamma K \sinh(\gamma t + \varepsilon) = \frac{p}{m}, \qquad \dot{p} = m\gamma^2 K \cosh(\gamma t + \varepsilon) = aq.$$

In order to relate this solution to the one given in the text, we write cosh in terms of exponentials: thus

$$q(t) = \tfrac{1}{2}K(e^{(\gamma t+\varepsilon)} + e^{-(\gamma t+\varepsilon)}) = \tfrac{1}{2}Ke^{\varepsilon}e^{\gamma t} + \tfrac{1}{2}Ke^{-\varepsilon}e^{-\gamma t}.$$

Comparing this with the expression

$$q(t) = A_+ e^{\gamma t} + A_- e^{-\gamma t},$$

we see that we must have

$$A_+ = \tfrac{1}{2}Ke^{\varepsilon}, \qquad A_- = \tfrac{1}{2}Ke^{-\varepsilon},$$

whence

$$K^2 = 4A_+A_-.$$

Thus the given representation of the phase curve is possible only when $A_+A_- > 0$ (a result that is also clear from the fact that $\cosh(\gamma t + \varepsilon) \geq 1$ always); but when this inequality does hold we can express the phase curve in the given form with

$$K = \pm 2\sqrt{A_+A_-}, \qquad \varepsilon = \ln\sqrt{\frac{A_+}{A_-}}.$$

In this case A_+ and A_- have the same sign, so we are dealing with the left and right quadrants in Figure S4.2. The choice of sign in K determines the branch of the hyperbola, the plus sign corresponding to the right-hand branch (like curve B in Figure 4.9, page 103) and the minus sign to the left-hand branch (curve D). Note that the argument of the logarithm is positive, since A_+ and A_- have the same sign, and so ε is well defined. The energy in this case is negative, and K is given in terms of the energy by

$$K = \sqrt{2\frac{|E|}{m\gamma^2}}.$$

To treat the case when $A_+A_- < 0$ we must clearly introduce a minus sign into the expression for $q(t)$, which suggests replacing the cosh by a sinh. So let us try

$$q(t) = K\sinh(\gamma t + \varepsilon).$$

Then

$$\dot{q} = \gamma K\cosh(\gamma t + \varepsilon) = \frac{p}{m}, \qquad \dot{p} = m\gamma^2 K\sinh(\gamma t + \varepsilon) = aq,$$

and once again we have a solution of Hamilton's equations. Now

$$q(t) = \tfrac{1}{2}K(e^{(\gamma t+\varepsilon)} - e^{-(\gamma t+\varepsilon)}) = \tfrac{1}{2}Ke^{\varepsilon}e^{\gamma t} - \tfrac{1}{2}Ke^{-\varepsilon}e^{-\gamma t}.$$

Comparing this with the expression

$$q(t) = A_+e^{\gamma t} + A_-e^{-\gamma t},$$

we see that we must now have

$$A_+ = \tfrac{1}{2}Ke^{\varepsilon}, \qquad A_- = -\tfrac{1}{2}Ke^{-\varepsilon},$$

whence

$$K^2 = -4A_+A_-.$$

Thus the new representation of the phase curve requires that $A_+A_- < 0$, as expected. When this inequality holds, the phase curve can always be represented in terms of hyperbolic functions as given above, with

$$K = \pm 2\sqrt{-A_+A_-}, \qquad \varepsilon = \ln\sqrt{-\frac{A_+}{A_-}}.$$

We are now concerned with the upper and lower quadrants in Figure S4.2. The choice of sign for K once again determines the branch of the hyperbola, the plus sign corresponding to the upper branch (like curve A in Figure 4.9) and the minus sign to the lower branch (curve C). The relation between K and the energy is the same as before,

$$K = \sqrt{2\frac{E}{m\gamma^2}},$$

but the energy is now positive.

On a branch separatrix either $A_+ = 0$ or $A_- = 0$, and this limit can be obtained from the above analysis only by letting $\varepsilon \to -\infty$ ($A_+ = 0$), or $\varepsilon \to \infty$ ($A_- = 0$), and choosing K such that A_- or A_+, respectively, is finite. In other words, this representation of the solution does not conveniently give the motion on the separatrix.

Solution 4.20

(i) For the linear system to be Hamiltonian we need

$$\dot{q} = aq + bp = \frac{\partial H}{\partial p} \quad\Longrightarrow\quad \frac{\partial^2 H}{\partial q\partial p} = a,$$

$$\dot{p} = cq + dp = -\frac{\partial H}{\partial q} \quad\Longrightarrow\quad \frac{\partial^2 H}{\partial p\partial q} = -d,$$

so we must have $a + d = 0$. On the other hand, the most general Hamiltonian giving rise to a set of linear equations is a quadratic one, say

$$H(q,p) = \tfrac{1}{2}Aq^2 + Bqp + \tfrac{1}{2}Cp^2 \tag{S4.2}$$

for some constants A, B and C. These give the equations of motion

$$\dot{q} = \frac{\partial H}{\partial p} = Bq + Cp, \qquad \dot{p} = -\frac{\partial H}{\partial q} = -Aq - Bp,$$

in which, in comparison with the given equations, we have $a = B = -d$.

The same conclusions follow on using the mixed derivative test (Equation 4.15 on page 93).

(ii) The eigenvalues satisfy the equations $\lambda^2 + \det \mathbf{A} = 0$ since $\operatorname{tr} \mathbf{A} = 0$, so that

$$\begin{aligned} \lambda &= \pm\sqrt{-\det \mathbf{A}}, \qquad &\det \mathbf{A} < 0, \\ \lambda &= \pm i\sqrt{\det \mathbf{A}}, \qquad &\det \mathbf{A} > 0. \end{aligned}$$

The fixed point of a linear Hamiltonian system is

- a saddle if $\det \mathbf{A} < 0$,
- a centre if $\det \mathbf{A} > 0$.

The nature of the fixed point depends only upon the sign of $\det \mathbf{A}$, and $\det \mathbf{A}$ is given in terms of the coefficients by

$$\det \mathbf{A} = -(a^2 + bc), \qquad \text{since} \quad d = -a.$$

Thus the fixed point is

- a saddle if $a^2 + bc > 0$,
- a centre if $a^2 + bc < 0$.

(It might perhaps be useful to express the conditions in terms of the coefficients of the Hamiltonian function also. The fixed point is

- a saddle if $B^2 - AC > 0$,
- a centre if $B^2 - AC < 0$.)

(iii) The fixed point of a linear oscillator is a centre; the fixed point for a linear repulsive force is a saddle. No other type of fixed point is possible for a linear Hamiltonian system.

Solution 4.21

(i) At a fixed point,

$$\dot{q} = \frac{\partial H}{\partial p} = 0, \qquad \dot{p} = -\frac{\partial H}{\partial q} = 0,$$

so a fixed point is a stationary point of H.

(ii) Since $v_q(q,p) = \partial H/\partial p$ and $v_p(q,p) = -\partial H/\partial q$, the matrix of the linearised equations is

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$$\mathbf{A} = \begin{bmatrix} \dfrac{\partial v_q}{\partial q} & \dfrac{\partial v_q}{\partial p} \\ \dfrac{\partial v_p}{\partial q} & \dfrac{\partial v_p}{\partial p} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial^2 H}{\partial q \partial p}(q_0, p_0) & \dfrac{\partial^2 H}{\partial p^2}(q_0, p_0) \\ -\dfrac{\partial^2 H}{\partial q^2}(q_0, p_0) & -\dfrac{\partial^2 H}{\partial q \partial p}(q_0, p_0) \end{bmatrix}.$$

Since $\operatorname{tr} \mathbf{A} = 0$ it follows from the previous exercise that the linearised system is also Hamiltonian, and its fixed point is either a saddle or a centre.

Solution 4.22

This question can be answered either by appealing to the general theory of stationary points of functions of two variables or from first principles; the latter is more instructive. We assume that a change of variables has been made to bring the stationary point to the origin. The Hamiltonian for the linearised system is given by Equation S4.2, where

$$A = \frac{\partial^2 H}{\partial q^2}, \qquad B = \frac{\partial^2 H}{\partial q \partial p}, \qquad C = \frac{\partial^2 H}{\partial p^2},$$

the second partial derivatives being evaluated at the fixed point. If $A \neq 0$ this can be recast in the form

$$\begin{aligned} H(q,p) &\simeq \frac{1}{2A}\left(A^2 q^2 + 2ABqp + ACp^2\right) \\ &= \frac{1}{2A}\left((Aq + Bp)^2 + (AC - B^2)p^2\right), \end{aligned}$$

on completing the square. The form of the graph of this function, which depends on the signs of the coefficients, determines the nature of the stationary point of the original Hamiltonian and the nature of the fixed point of the linearised system.

Thus if $AC - B^2 > 0$ the Hamiltonian has a local minimum if $A > 0$ or a local maximum if $A < 0$, so the fixed point is a centre. If $AC - B^2 < 0$ the stationary point is a saddle point and the fixed point is a saddle, whatever the sign of A.

If $A = 0$ but $C \neq 0$ then

$$H(q,p) \simeq Bqp + \tfrac{1}{2}Cp^2 = \frac{1}{2C}\left(-B^2q^2 + (Bq + Cp)^2\right),$$

so that the contours of H are hyperbolas (approximately) and the stationary point (fixed point) is a saddle. If both A and C are zero then $H(q,p) \simeq Bqp$, giving a saddle.

These results correspond precisely to the usual classification of a stationary point. That classification goes as follows:

$AC - B^2 < 0$: saddle point;
$AC - B^2 > 0$ and $A < 0$: maximum;
$AC - B^2 > 0$ and $A > 0$: minimum;
$AC - B^2 = 0$: no information.

The corresponding results on the nature of the fixed point of a linear system are given in the solution to Exercise 4.20(ii). We noted that the fixed point is

a saddle if $AC - B^2 < 0$,
a centre if $AC - B^2 > 0$.

Solution 4.23

In the case $H(q,p) = p^2/2m + V(q)$ the fixed points occur where

$$p = 0, \qquad V'(q) = 0.$$

Note that the section of the graph of the Hamiltonian function by a vertical plane through the fixed point parallel to the p-axis is just the graph of the function $p^2/2m +$ constant; thus the Hamiltonian has a minimum in the p direction, and therefore the fixed point can be only a saddle or a minimum. At a minimum of the potential $V(q)$ the Hamiltonian also has a minimum, so such a point is a centre. At a maximum of the potential the Hamiltonian has a saddle point, so such a point is a saddle of the Hamiltonian system.

Solution 4.24

The potential $V(q) = -\frac{1}{2}\gamma^2q^2 + \frac{1}{3}Bq^3$ is sketched in the figure.

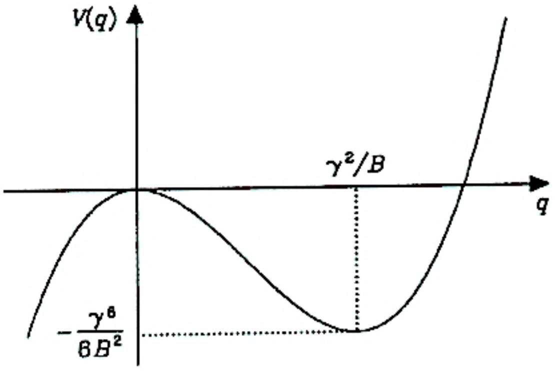


Figure S4.3 Graph of the potential $V(q) = -\frac{1}{2}\gamma^2q^2 + \frac{1}{3}Bq^3$.

The potential has a local maximum at $q = 0$ and a minimum at $q = \gamma^2/B$. There is thus a fixed point at $(0,0)$, which is a saddle, and a separatrix of energy $E = 0$ through this fixed point. The value of the energy at the other fixed point is $-\gamma^6/6B^2$. Thus bounded motion is possible for $-\gamma^6/6B^2 < E < 0$. For $E < -\gamma^6/6B^2$ or $E > 0$ only unbounded motion is possible. A sketch of the upper half of the phase plane is shown in Figure S4.4.

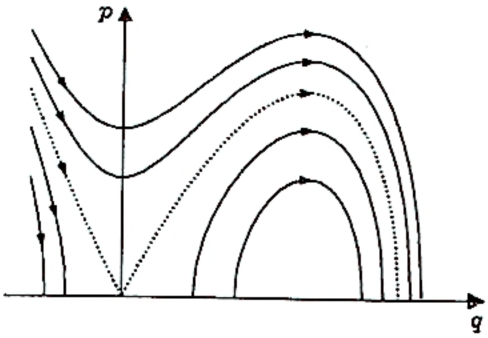


Figure S4.4 Sketch of the phase curves for the potential $V(q) = -\gamma^2 q^2/2 + Bq^3/3$. For $p < 0$ the phase curves are obtained by a reflection in the q-axis. The separatrix, with energy $E = 0$, is depicted by the dotted line.

Solution 4.25

On putting $A' = -A$, the potential of the example becomes

$$V(q) = \tfrac{1}{2}\omega^2 q^2 - \tfrac{1}{3}A' q^3.$$

Putting $q' = -q$, on the other hand, gives

$$V(q') = \tfrac{1}{2}\omega^2 (q')^2 - \tfrac{1}{3}A(q')^3.$$

Thus changing the sign of A is equivalent to changing the sign of q, and the phase curves for $A < 0$ are obtained from those for $A > 0$ by reflection in the p-axis.

Similarly for the potential $V(q) = -\frac{1}{2}\gamma^2 q^2 + \frac{1}{3}Bq^3$, $B < 0$.

Solution 4.26

If $c > 0$ then $V(q) \to \pm\infty$ as $q \to \pm\infty$, and since $V(q)$ is a cubic it has either two or no real stationary points, which are at the roots of

$$\frac{dV}{dq} = a + bq + cq^2 = 0.$$

This equation has two real roots if $b^2 > 4ac$, in which case $V(q)$ is shaped like the potential of Exercise 4.24 (see Figure S4.3), having a minimum at some point q_- and a maximum at q_+ with $q_+ < q_-$. If $b^2 < 4ac$ then $V(q)$ is monotonic increasing and there is no bounded motion. If $b^2 = 4ac$ then $V'(q) = 0$ has a double root and the fixed point is not simple.

The case $c < 0$ can be dealt with by noting that

$$V(-q) = (-a)q + \tfrac{1}{2}bq^2 + \tfrac{1}{3}(-c)q^3,$$

which comes within the scope of the cases already considered.

Solution 4.27

The Hamiltonian is $H(q,p) = \frac{1}{2}(p^2 - q^3) = E$, so for the initial conditions given, $E = 0$ and $p^2 = q^3$. Hamilton's equations then give

$$\dot{q} = \frac{\partial H}{\partial p} = p = q^{3/2}.$$

This equation can be integrated directly by separating the variables,

$$\int_1^q \frac{dq}{q^{3/2}} = \int_0^t dt \qquad \text{or} \qquad q(t) = \left(\frac{2}{2-t}\right)^2, \qquad 0 \le t < 2,$$

showing that the motion terminates at $t = 2$.

Solution 4.28

The Hamiltonian is $H(q,p) = p^2/2m - Aq^\gamma = E$, so for the initial conditions given, $E = 0$ and $p^2 = 2mAq^\gamma$. Hamilton's equations then give

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m} = q^{\gamma/2}\sqrt{\frac{2A}{m}}.$$

This equation can be integrated directly by separating the variables,

$$\int_1^q \frac{dq}{q^{\gamma/2}} = \sqrt{\frac{2A}{m}}\int_0^t dt \quad \text{or} \quad q^\beta - 1 = \beta t\sqrt{\frac{2A}{m}}, \quad \beta = 1 - \gamma/2. \qquad \text{(S4.3)}$$

If $\gamma < 2$ then $\beta > 0$ and

$$q(t) = \left(1 + \beta t\sqrt{\frac{2A}{m}}\right)^{1/\beta},$$

and $q(t)$ is finite for all positive, finite t. If $\gamma > 2$ then $\beta < 0$ and

$$q(t) = \left(1 - \delta t\sqrt{\frac{2A}{m}}\right)^{-1/\delta}, \qquad \delta = -\beta = \gamma/2 - 1 > 0,$$

so that at $t = \sqrt{m/2A}/\delta$, $q(t)$ becomes infinite and the motion terminates.

If $\gamma = 2$ Equation S4.3 becomes

$$\ln q = t\sqrt{\frac{2A}{m}} \qquad \text{or} \qquad q(t) = \exp\left(t\sqrt{\frac{2A}{m}}\right).$$

This is the case of the linear repulsive force dealt with in Example 4.3 (page 103).

Solution 4.29

(i) The potential $V(q) = Aq^2\exp(-q^2)$ is non-negative, zero at $q = 0$ and approaches zero as $q \to \pm\infty$; it must therefore have at least two maxima and one minimum. The stationary points are at the roots of

$$\frac{dV}{dq} = 2Aq\left(1 - q^2\right)\exp(-q^2),$$

which are $q = 0$ and $q = \pm 1$. The above discussion shows that the only minimum is at $q = 0$ and that the maxima are at $q = \pm 1$, with value A/e. A sketch of the potential (for $A = 1$) is given in the lower half of Figure S4.5.

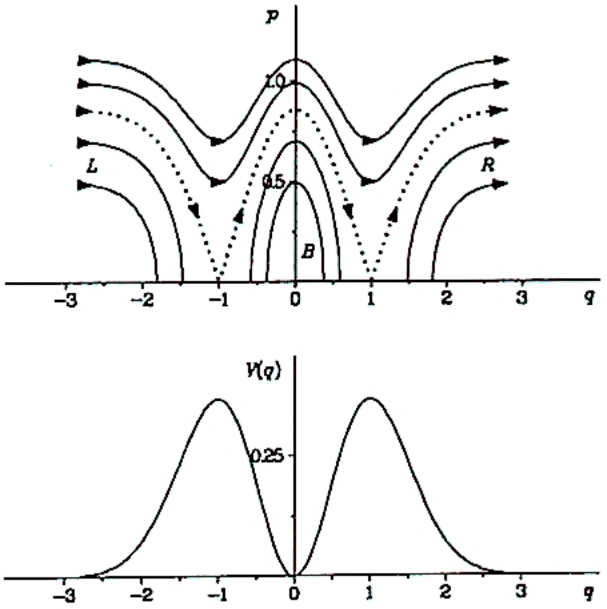


Figure S4.5 The lower diagram shows the potential $V(q) = q^2\exp(-q^2)$ and the upper diagram some of the phase curves, the dotted line being the separatrix which, in this case, has six different branches.

From this sketch we see that the energy must be positive and that four types of motion are possible. For energies $E > A/e = 0.368A$ all motion has sufficient energy to pass over the potential maxima at $q = \pm 1$, so these phase curves, in the upper half phase plane, represent motion starting from the far left, $q \ll 0$, and passing to the far right, $q \gg 0$. There are similar phase curves in the lower half phase plane, but these represent motion in the reverse direction, from right to left.

For energies $E < A/\epsilon$ three types of motion are possible. There is bounded motion with $-q_1(E) \leq q(t) \leq q_1(E)$, where $q_1(E)$ is the smallest positive root of $q^2 \exp(-q^2) = E/A$; this is exemplified by the curve B in the upper half of Figure S4.5. There is also motion which starts from the far right with $p < 0$, but with insufficient energy to surmount the barrier, which is reflected at $q_2(E)$, the largest positive root of $q^2 \exp(-q^2) = E/A$, and then returns to the right with $p > 0$; this motion is shown by the curve R in the figure. Similarly, the curve L shows motion starting from the left, with $p > 0$, which is reflected at $-q_2(E)$.

The separatrix, s, shown by the dotted lines, has six branches, three in the upper half-plane and three in the lower half-plane. The central branch divides the bounded from the unbounded motion and the two outer branches the motion that can pass over the barrier from that which cannot. The separatrix equation is

$$p_s(q) = \sqrt{\frac{2A}{e}}\left(1 - q^2 \exp(1 - q^2)\right)^{1/2}.$$

Notice that because the Hamiltonian is an even function of p the phase curves for $p < 0$ are obtained from those for $p > 0$ by a reflection in the q-axis. For this reason we show only the upper half phase plane.

(ii) The potential $V(q) = q^{-4} - 2q^{-2}$ is dominated by the term q^{-4} as $q \to 0$, so is positive and large for q small; for large q, $V(q) \simeq -2q^{-2}$, so it approaches zero from below. The potential has only one real root for $q > 0$, at $q = 1/\sqrt{2}$, and one stationary point at $q = 1$, given by the zero of

$$\frac{dV}{dq} = \frac{4}{q^5}\left(q^2 - 1\right) = 0, \qquad q > 0.$$

The value of $V(q)$ at this stationary point is -1. From the above discussion it is clear that the stationary point is a minimum. This information is sufficient to sketch the graph of the potential, which is shown in the lower half of Figure S4.6.

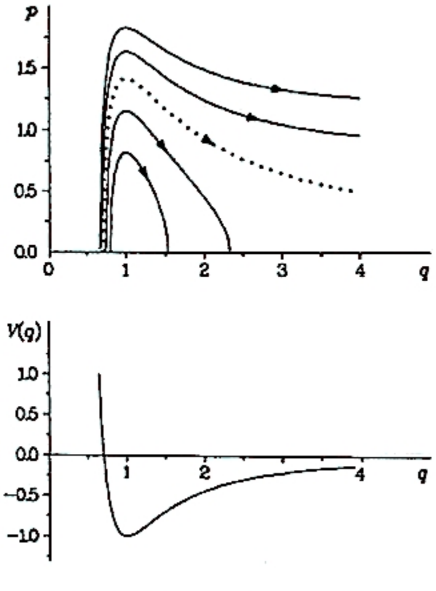


Figure S4.6 The lower diagram shows the potential $V(q) = q^{-4} - 2q^{-2}$ and the upper diagram some of the phase curves, the dotted line being a separatrix which, in this case, approaches the q-axis only as $q \to \infty$.

From this sketch we see that for energies $-1 < E < 0$ the motion is bounded, with turning points given by the roots of $V(q) = E$, that is,

$$q = \left(1 \pm \sqrt{1+E}\right)^{-1/2};$$

as $E \to 0$ these tend to $1/\sqrt{2}$ and ∞. Two such phase curves, at energies $E = -\frac{1}{3}$ and $-\frac{2}{3}$, are shown in the upper half of Figure S4.6.

For $E > 0$ the motion is unbounded and there is only one real turning point, at

$$q = \left(1 + \sqrt{1+E}\right)^{-1/2}.$$

For large q, where $V(q) \simeq 0$, the phase curves are given approximately by $p = \pm\sqrt{2E}$. Two such curves are shown in the figure.

The $E = 0$ phase curve is a separatrix, shown by the dotted line in the figure, since it divides the phase plane into two invariant regions supporting qualitatively different types of motion. It has the equation

$$\begin{aligned} p_s(q) &= \frac{2}{q}\left(1 - \frac{1}{2q^2}\right)^{1/2} \\ &\simeq \frac{2}{q}, \qquad q \gg 1. \end{aligned}$$

Solution 4.30

(i)

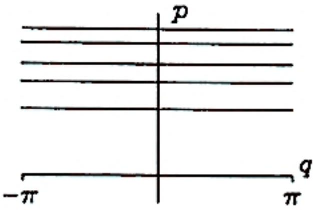


Figure S4.7 Hamiltonian contours for $H = p^2/2$ in the periodic phase plane.

The equally-spaced energy contours are shown in Figure S4.7. The spacing between them decreases with increasing E, so the speed of the particle, which is inversely proportional to the contour spacing, increases. We should therefore expect the period to decrease with increasing energy.

(ii) The period $T(E)$ is the time taken for q to increase by 2π, so from Equation 4.39 (page 116), since $\mu = 1$,

$$2\pi = p_0 T, \qquad \text{where} \quad E = p_0^2/2,$$

or

$$T(E) = \frac{2\pi}{\sqrt{2E}}.$$

Solution 4.31

Near $q = 0$ we can use the approximate Hamiltonian given in Equation 4.46 (page 118), with the equations of motion

$$\dot{q} = p/ml^2, \qquad \dot{p} = -Flq.$$

Together these give

$$\ddot{q} + \frac{F}{ml}q = 0,$$

which is the equation for simple harmonic motion of angular frequency ω, where $\omega^2 = F/ml$, and period

$$T = 2\pi\sqrt{\frac{ml}{F}}.$$

Solution 4.32

(i) The contour equation for the energy E is

$$p(q,E) = \pm\sqrt{2ml^2(E + Fl\cos q)},$$

and since the separatrix energy is $E_s = Fl$, we may use the relation $\cos q = 2\cos^2 \frac{1}{2}q - 1$ to express the equation of the separatrix in the form

$$p_s(q) = \pm\sqrt{2mFl^3(1+\cos q)} = \pm 2\sqrt{mFl^3}\cos\tfrac{1}{2}q.$$

(ii) The area enclosed between the q-axis and the upper half of the separatrix is $\frac{1}{2}A_s$ and is given by

$$\begin{aligned}\tfrac{1}{2}A_s = \int_{-\pi}^{\pi} dq\, p_s(q) &= 2\sqrt{mFl^3}\int_{-\pi}^{\pi} dq\, \cos\tfrac{1}{2}q \\ &= 8\sqrt{mFl^3}\int_0^{\pi/2} dx\, \cos x \\ &= 8\sqrt{mFl^3}.\end{aligned}$$

Hence the result.

Solution 4.33

Since $V(q) = 0$, Equation 4.54 gives

$$T(E) = \sqrt{\frac{m}{2}}\int_0^{2\pi}\frac{dq}{\sqrt{E}} = 2\pi\sqrt{\frac{m}{2E}},$$

which is the same as the result obtained in Exercise 4.30 if $m = 1$.

Solution 4.34

If $q(0) = q_0$ then (as in the text) the time taken for q to increase from q_0 to $q_0 + 2\pi$ is

$$\begin{aligned}T &= \int_{q_0}^{q_0+2\pi}\frac{dq}{\dot q} \\ &= \int_0^{2\pi}\frac{dq}{\dot q} + \int_{2\pi}^{2\pi+q_0}\frac{dq}{\dot q} - \int_0^{q_0}\frac{dq}{\dot q}.\end{aligned}$$

But as $\dot q(q,E)$ is a periodic function of q, that is

$$\dot q(2\pi + x, E) = \dot q(x,E) \qquad \text{for all } x,$$

we can change the variable in the last integral from q to $x = 2\pi + q$ to see that it cancels the second integral to give the desired result.

Solution 4.35

In the case of the potential $V(q) = -Fl\cos q$ with $E > Fl$ the period is

$$\begin{aligned}T(E) &= \sqrt{\frac{m}{2}}\int_{-\pi}^{\pi}\frac{dq}{\sqrt{E + Fl\cos q}} \\ &= \sqrt{2m}\int_0^{\pi}\frac{dq}{\sqrt{E+Fl\cos q}} \\ &= \sqrt{2m}\int_0^{\pi}\frac{dq}{\sqrt{E + Fl - 2Fl\sin^2\frac{1}{2}q}} \\ &= \sqrt{\frac{8m}{E+Fl}}\int_0^{\pi/2}\frac{dx}{\sqrt{1-k^2\sin^2 x}}, \qquad k^2 = \frac{2Fl}{E+Fl} < 1,\end{aligned}$$

where we have used the half-angle formula and then put $q = 2x$.

Solution 4.36

If $\dot{q} < 0$ then $q(t)$ is decreasing and (as in the text)

$$T(E) = \int_0^{-2\pi} \frac{dq}{\dot{q}}, \qquad \dot{q} < 0.$$

But since $\dot{q} = -\sqrt{(2/m)(E - V(q))}$ this gives the required equation. Now put $q' = 2\pi + q$ and use the periodicity of $V(q)$:

$$T(E) = -\sqrt{\frac{m}{2}} \int_{2\pi}^{0} \frac{dq'}{\sqrt{E - V(q')}} = \sqrt{\frac{m}{2}} \int_0^{2\pi} \frac{dq}{\sqrt{E - V(q)}}.$$

Solution 4.37

For the linear oscillator the potential is $V(q) = \frac{1}{2}kq^2$, so the turning points are $q = \pm\sqrt{2E/k}$ and the period is

$$T(E) = \sqrt{2m} \int_{-\sqrt{2E/k}}^{\sqrt{2E/k}} \frac{dq}{\sqrt{E - \frac{1}{2}kq^2}}.$$

In order to evaluate this integral make the substitution $q = \sqrt{2E/k}\sin\phi$, to give

$$T(E) = 2\pi\sqrt{\frac{m}{k}}.$$

Solution 4.38

The equations of motion are

$$\dot{q} = \frac{p}{m}, \quad \dot{p} = -kq, \qquad \text{or} \qquad m\ddot{q} + kq = 0,$$

which have the general solution

$$q(t) = A\cos\omega t + B\sin\omega t, \qquad p(t) = -m\omega A\sin\omega t + m\omega B\cos\omega t,$$

where $\omega^2 = k/m$. On fitting the initial conditions we see that $A = Q$ and $P = m\omega B$, giving the required result. It is clear that $q(t)$ and $p(t)$ depend linearly on Q and P, and in fact the equations above can be written in the matrix form

$$\begin{bmatrix} q(t) \\ p(t) \end{bmatrix} = \begin{bmatrix} \cos\omega t & \sin\omega t/m\omega \\ -m\omega\sin\omega t & \cos\omega t \end{bmatrix} \begin{bmatrix} Q \\ P \end{bmatrix}.$$

The determinant of the matrix is

$$\cos^2\omega t - (-\sin^2\omega t) = 1,$$

showing that the transformation $(Q, P) \mapsto (q(t), p(t))$ is area-preserving.

Solution 4.39

(i) The new coordinates are just the time along a trajectory and the energy, so (q, p) are to be regarded as functions of (t, E). The Jacobian of the transformation $(q, p) \mapsto (t, E)$ is just

$$J = \begin{vmatrix} \dfrac{\partial q}{\partial t} & \dfrac{\partial p}{\partial t} \\ \dfrac{\partial q}{\partial E} & \dfrac{\partial p}{\partial E} \end{vmatrix}.$$

Now Hamilton's equations give $\partial q/\partial t = \partial H/\partial p$ and $\partial p/\partial t = -\partial H/\partial q$, so we may write

$$J = \frac{\partial H}{\partial p}\frac{\partial p}{\partial E} + \frac{\partial H}{\partial q}\frac{\partial q}{\partial E}.$$

But $H(q,p) = E$, so differentiation gives

$$\frac{\partial H}{\partial p}\frac{\partial p}{\partial E} + \frac{\partial H}{\partial q}\frac{\partial q}{\partial E} = 1,$$

which means that

$$J = 1,$$

as required.

(ii) Because the system is conservative, $E =$ constant along any phase curve, so in the (t, E)-representation the motion is just the time-translation $(t, E) \mapsto (t + T, E)$.

In the (t, E)-representation the flow is area-preserving, being just a translation, and since the transformation $(q, p) \mapsto (t, E)$ was found to be area-preserving in part (i) it follows that the flow in the (q, p)-representation is also area-preserving.

END OF BLOCK EXERCISES

Exercises on Unit 2

Exercise 2.1 S

What is the phase space of a dynamical system? What is the order of a dynamical system?

(i) Consider a biological system consisting of two species, one of which preys on the other. The size of the population of the predator is p, while that of the prey is q. In a model of how these population sizes change, the following differential equations are proposed:

$$\frac{dp}{dt} = Apq - Bp, \qquad \frac{dq}{dt} = -Cpq + Dq,$$

where A, B, C and D are positive constants. What would you take for the phase space of this system?

(ii) Consider, on the other hand, a particle moving on a horizontal frictionless plane table subject to no horizontal forces. Why is the plane of the table not a suitable phase space for this system? What is the order of the system?

Exercise 2.2 S

For each of the following equations of motion, specify the order of the corresponding dynamical system, and state whether or not the system is autonomous.

(i) $\dot{x} = y + x(1 - y^2), \quad \dot{y} = -x + y(1 - x^2)$

(ii) $\dfrac{du}{dx} = x^2 + u^2$

(iii) $\dfrac{d^2y}{dx^2} = y^2$

(iv) $\dot{q} = p, \quad \ddot{p} = -q$

(v) $\ddot{q} + 2\alpha\omega\dot{q} + \omega^2 q = \omega^2(a\cos\Omega t + b\sin\Omega t + c)$,
where α, ω, Ω, a, b and c are constants.

Exercise 2.3 S

A point moves in the plane in such a way that its velocity field is the gradient of the scalar field $\frac{1}{2}(x^2 + y^2)$. Write down its equations of motion.

Exercise 2.4 S

The purpose of this question is to prove the rather unsurprising result that the faster a system goes, the shorter the time it takes to get from A to B.

Consider two first-order autonomous systems with velocity functions $v_1(x)$ and $v_2(x)$. Suppose that, for some constants A and B with $A < B$,

$$0 < v_1(x) \leq v_2(x) \qquad \text{when} \quad A \leq x \leq B,$$

and that $v_1(B)$, $v_2(B)$ are finite. By considering the solution of the equations of motion by separation of variables, show that, if T_i $(i = 1, 2)$ is the time that the system with velocity function v_i takes to go from state $x = A$ to state $x = B$, then

$$T_2 \leq T_1.$$

Exercise 2.5 S

Describe how the phase diagram of an autonomous first-order system is drawn, and how it is used.

Exercise 2.6 S

(i) Define the terms fixed point, invariant set and elementary invariant set of an autonomous first-order system. Find the fixed points and invariant sets of the system whose velocity field v is given by

$$v(x) = x(1-x).$$

(ii) When is a fixed point simple? Which of the fixed points in the example above are simple?

(iii) When is a simple fixed point stable, and when unstable? Classify the simple fixed points in the example above.

(iv) Sketch a phase diagram for the system with $v(x) = x(1-x)$.

Exercise 2.7 S

A spherical drop of fluid loses mass by evaporation at a rate proportional to its surface area. Determine the differential equation for the radius $R(t)$ as a function of the time t, and describe the change of radius qualitatively.

Exercise 2.8 S

Discuss the nature of the motion and give the invariant sets of the systems with velocity functions

(i) $v(x) = x\sin x$, (ii) $v(x) = x\cos x$.

Exercise 2.9 S

Determine explicit expressions for the flow $\phi_t(x)$ in the cases

(i) $\dot{x} = bx$, (ii) $\dot{x} = bx^2$, (iii) $\dot{x} = x - x^2$, (iv) $\dot{x} = x - x^3$,

where b is a constant.

Exercise 2.10 S

How many essentially different types of phase portraits can occur for an autonomous first-order system with three simple fixed points? How is your answer changed if the fixed points are not necessarily simple?

Exercise 2.11 S

Consider the system with the cubic velocity function $\overline{v}(x) = x^3 + \varepsilon x$. Describe how the number of its fixed points, and their nature, depend on the value of ε.

Exercise 2.12 S

A system is said to be structurally stable if its flow retains its structure under any sufficiently small perturbation of its velocity field; otherwise it is structurally unstable. Give examples of structurally stable and structurally unstable systems.

Exercise 2.13 S

Show that the motion of the system

$$\dot{x} = x^\alpha, \qquad x > 0,$$

terminates (that is, reaches infinity in a finite time) for any α greater than 1.

Suppose that the function $v(x)$ is well-behaved for all $x > 0$, and that there are positive numbers k and ε such that $v(x) > kx^{1+\varepsilon}$ for all sufficiently large x. Show that for the system with velocity function $v(x)$ any motion for which the initial state x_0 is sufficiently large terminates. [The result of Exercise 2.4 above will be useful.]

Exercise 2.14 S

For each of the following velocity functions, give a qualitative description of the motion of the system with that velocity function, and obtain expressions for the motion in the vicinity of any fixed point.

(i) $v(x) = a\sqrt{x-b}, \quad a > 0, \quad x \geq b.$

(ii) $v(x) = a\sqrt{b-x}, \quad a > 0, \quad x \leq b.$

Exercise 2.15 S

(i) Show that, if $f(x)$ is a well-behaved function such that $f(b) = 0$ and $f'(b) < 0$, so that the Taylor expansion of $f(x)$ about b may be written

$$f(x) = -a^2(x-b) + O((x-b)^2)$$

for some real, non-zero constant a, then the motion defined by

$$v(x) = \sqrt{f(x)}$$

terminates at $x = b$.

(ii) Describe the motion with velocity function

$$w(x) = \sqrt{g(x)},$$

where $g(x)$ is a polynomial of degree N with N different real roots.

Exercise 2.16 S

Nitric oxide, NO, and oxygen, O_2, react to form nitric dioxide, NO_2, as follows:

$$2NO + O_2 \rightarrow 2NO_2.$$

It is found that the concentration $C(t)$ of NO_2 at time t satisfies the differential equation

$$\frac{dC}{dt} = k(\alpha - C)^2(2\beta - C), \qquad C \geq 0, \quad C(0) = 0,$$

where α and β are the initial concentrations of NO and O_2 respectively, both being greater than zero, and k is a positive constant determined by the reaction but independent of α and β. Use graphical methods to describe the qualitative behaviour of the NO_2 concentration, assuming that initially it is zero, distinguishing between the cases $\alpha < 2\beta$ and $\alpha > 2\beta$.

Exercise 2.17 E

Sulphur dioxide reacts with oxygen to form SO_3 as follows:

$$2SO_2 + O_2 \rightarrow 2SO_3.$$

The concentration $C(t)$ of SO_3 satisfies the equation

$$\frac{dC}{dt} = k(\alpha - C)C^{-1/2}, \qquad C \geq 0, \quad C(0) = 0,$$

where k is a positive constant for the reaction and $\alpha > 0$ is the initial concentration of SO_2. Discuss the change in concentration $C(t)$ for small times and for large times.

Exercise 2.18 S

A point moves round a circle in such a way that

$$\dot{\theta} = \sqrt{k + \cos\theta},$$

where θ is the angle its position vector makes with a fixed radius of the circle and k is a constant such that $k \geq -1$. For what values of k is this motion a rotation? Write down, but do not evaluate, an integral which gives the period when the motion is a rotation.

Exercise 2.19 S

With air resistance proportional to the square of the speed, the speed $V(t)$, a positive quantity, of a falling body is given by the equation

$$\frac{dV}{dt} = g - kV^2, \qquad g > 0, \quad k > 0, \quad V(t) \geq 0.$$

Describe the change of $V(t)$ with time, for arbitrary initial non-negative values of $V(0)$. What is the limiting behaviour as $t \to \infty$?

Exercise 2.20 E

Consider a first-order non-autonomous system with a velocity function $v(x, t)$ which is homogeneous of degree zero, that is, which satisfies

$$v(ax, at) = v(x, t) \qquad \text{for all non-zero constants } a.$$

Show that by the change of variables $x = tu$ and $t = e^\tau$ the system $\dot{x} = v(x, t)$ reduces to the autonomous system

$$\frac{du}{d\tau} = v(u, 1) - u.$$

Show that a fixed point of this autonomous system corresponds to uniform motion $x = kt$, where k is some constant, of the original system. Discuss the nature of the motion as $t \to \infty$ of the original system when the fixed point of the autonomous system is simple and stable.

Exercise 2.21 E

In general the motions of non-autonomous systems are difficult to find and to understand, but there is a class of such systems which have properties as easy to understand as those of autonomous systems. These are systems which have linear equations of motion with periodic coefficients, that is, equations of the form

$$\frac{dx}{dt} = xv(t),$$

where $v(t)$ is an integrable, periodic function of period T:

$$v(t + T) = v(t) \qquad \text{for all } t.$$

(i) Show that

$$x(b) = x(a) \exp\left(\int_a^b dt\, v(t)\right).$$

(ii) Show that, for any integer N,

$$x(NT + \tau) = \mu^N x(\tau), \qquad \mu = \exp\left(\int_0^T dt\, v(t)\right).$$

(iii) Determine conditions on $v(t)$ such that

- (a) $|x(t)|$ is unbounded as $t \to \infty$,
- (b) $x(t) \to 0$ as $t \to \infty$,
- (c) $x(t)$ is bounded as $t \to \infty$, but does not tend to 0.

(iv) Determine the values of α and T which produce bounded solutions of the system

$$\dot{x} = xv(t),$$

where $v(t)$ is the T-periodic function whose values on $[0, T]$ are given by

$$v(t) = \begin{cases} 1, & 0 < t \leq \alpha T, \\ -1, & \alpha T < t \leq T, \end{cases}$$

where α is a constant in the range $0 < \alpha < 1$.

Exercises on Unit 3

Exercise 3.1 S

Reduce the following equations to systems of first-order autonomous equations.

(i) $\dot{x} + xt + 1 = 0$

(ii) $\ddot{x} + x - x^3 = 0$

(iii) $\ddot{x} + 3\dot{x}^2 + 4x^2 = 0$

Exercise 3.2 E

Let $\phi_t(\mathbf{x})$ be the flow of a linear system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$.

By using the fact that the system is linear, show that $\phi_t(\mathbf{x})$ depends linearly on $\mathbf{x}$: that is, show that for any $\mathbf{x}$, $\mathbf{y}$ and any constants a, b,

$$\phi_t(a\mathbf{x} + b\mathbf{y}) = a\,\phi_t(\mathbf{x}) + b\,\phi_t(\mathbf{y}).$$

Deduce that there is a matrix-valued function of t – which we shall denote by $\hat{\mathbf{A}}(t)$ – such that

$$\phi_t(\mathbf{x}) = \hat{\mathbf{A}}(t)\mathbf{x}.$$

Use the fundamental property of flows to show that the following rules, similar to the usual rules for exponentials, are satisfied:

$$\hat{\mathbf{A}}(s)\hat{\mathbf{A}}(t) = \hat{\mathbf{A}}(s + t)$$

(where matrix multiplication is implied on the left-hand side) and

$$\hat{\mathbf{A}}(0) = \mathbf{I}$$

(where $\mathbf{I}$ is the identity matrix).

Show that the matrix function $\hat{\mathbf{A}}(t)$ satisfies the matrix differential equation

$$\frac{d\hat{\mathbf{A}}}{dt}(t) = \mathbf{A}\hat{\mathbf{A}}(t),$$

and derive the Taylor expansion

$$\hat{\mathbf{A}}(t) = \mathbf{I} + t\mathbf{A} + \frac{1}{2!}t^2\mathbf{A}^2 + \frac{1}{3!}t^3\mathbf{A}^3 + \cdots.$$

Exercise 3.3 S

Sketch the phase curves of the systems with the following velocity fields:

(i) $\mathbf{v}(\mathbf{x}) = (a, y)$,

(ii) $\mathbf{v}(\mathbf{x}) = (x, a)$

(where a is a positive constant),

(iii) $\mathbf{v}(\mathbf{x}) = (2y, 3x^2)$.

Exercise 3.4 S

Show that the system

$$\dot{x} = f(x, y), \qquad \dot{y} = g(x, y)$$

has the same phase curves as the system

$$\dot{x} = -f(x, y), \qquad \dot{y} = -g(x, y),$$

but traverses them in the opposite direction. Demonstrate this explicitly for the systems

(i) $\dot{x} = x, \quad \dot{y} = y,$

(ii) $\dot{x} = 2y, \quad \dot{y} = -3x^2.$

Exercise 3.5 E

For autonomous systems the phase curves can be obtained by eliminating the time and forming a single first-order differential equation for dy/dx (*Unit 3* Equation 3.10, page 51). Show that the same method can be used for non-autonomous equations of the form

$$\frac{dx}{dt} = f(t)v_x(x,y), \qquad \frac{dy}{dt} = f(t)v_y(x,y),$$

where $f(t)$ is a function of time only and $\mathbf{v}(\mathbf{x})$ is independent of the time. Show further that by defining a new independent variable, the system may be converted to a second-order autonomous system.

What would be the difference if f were a function of x and y as well as time?

Exercise 3.6 S

Define the following terms:

(i) fixed point,

(ii) attractor and repellor,

(iii) strongly stable point,

(iv) isolated fixed point,

(v) simple fixed point,

(vi) simple system.

Exercise 3.7 S

Show that the origin is a strongly stable fixed point of the system

$$\dot{x} = -x^3, \qquad \dot{y} = -y^3,$$

but not of the system

$$\dot{x} = -y^2, \qquad \dot{y} = -x^2.$$

Exercise 3.8 S

What does it mean to say that a region of phase space *contains* the motion from another region of phase space?

Exercise 3.9 S

Classify the following linear systems, and sketch their phase curves in the neighbourhood of the origin.

(i) $\dot{x} = x - 5y, \quad \dot{y} = x - y$

(ii) $\dot{x} = -4x + 2y, \quad \dot{y} = 3x - 2y$

(iii) $\dot{x} = x + y, \quad \dot{y} = x - 2y$

(iv) $\dot{x} = 2x + y, \quad \dot{y} = y - x$

Exercise 3.10 S

(i) Show that if $\mathrm{R}(\phi)$ is the matrix representing rotation through the angle ϕ then n applications of this matrix gives a rotation through $n\phi$,

$$\mathrm{R}(\phi)^n = \mathrm{R}(n\phi).$$

Show further that if $\mathrm{S}(\rho,\phi) = \rho\mathrm{R}(\phi)$, so that $\mathrm{S}(\rho,\phi)$ is a rotation followed by a dilation, then

$$\mathrm{S}(\rho,\phi)^n = \mathrm{S}(\rho^n, n\phi).$$

(ii) Show that

$$\text{if } \mathrm{B} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \quad \text{then} \quad \mathrm{B}^n = \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix}.$$

(iii) Show that if $\mathbf{B} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ then $\mathbf{B}^2 = \mathbf{P}^{-1}\mathbf{A}^2\mathbf{P}$, and in general

$$\mathbf{B}^n = \mathbf{P}^{-1}\mathbf{A}^n\mathbf{P}.$$

Hence write down an expression for $\mathbf{A}^n$ in the case where $\mathbf{A}$ has two distinct eigenvectors, real or complex.

Exercise 3.11 S

Determine all the possible types of the fixed point of the system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ when the matrix $\mathbf{A}$ takes the form

$$\mathbf{A} = \begin{bmatrix} a & b \\ -b & d \end{bmatrix},$$

with $a > 0$ and $d > 0$.

Exercise 3.12 S

What is a separable system?

Show that the system

$$\dot{x} = -\tfrac{3}{5}x + \tfrac{4}{5}y, \qquad \dot{y} = \tfrac{4}{5}x + \tfrac{3}{5}y$$

is separable in the variables (X, Y), where $x = X + 2Y$, $y = 2X - Y$. Using the new variables as coordinates, construct a phase diagram of the system.

Exercise 3.13 E

A cell population consisting of a mixture of 2-chromosome and 4-chromosome cells is described approximately by

$$\frac{dX}{dt} = (\lambda - \mu)X, \qquad \frac{dY}{dt} = \mu X + \nu Y,$$

where X is the number of 2-chromosome and Y the number of 4-chromosome cells; X and Y are both non-negative; λ, μ and ν are constants with $\lambda \neq \mu$ and $\nu \neq 0$.

Show that, whatever the values of λ, μ and ν, as $t \to \infty$ the proportion of 2-chromosome cells in the population tends to a level independent of the initial conditions. Find conditions on the values of these parameters which ensure that the limiting value of this ratio is non-zero, and find the limit in this case.

Exercise 3.14 S

(i) Is an isolated fixed point of a non-linear system necessarily simple?

(ii) Is a simple fixed point of a non-linear system necessarily isolated?

Exercise 3.15 S

For which simple fixed points does the linear classification always describe the local behaviour of the non-linear system?

Exercise 3.16 S

Find all the fixed points of the following non-linear systems, and where possible classify them. (Be especially careful in the final part.)

(i) $\dot{x} = 1 - xy, \quad \dot{y} = y(1 - x)$

(ii) $\dot{x} = ye^y, \quad \dot{y} = 1 - x^2$

(iii) $\dot{x} = 2xy - 6y - 8, \quad \dot{y} = y^2 - x^2$

(iv) $\dot{x} = 4 - 4x^2 - y^2, \quad \dot{y} = 3xy$

(v) $\dot{x} = \sin(x + y), \quad \dot{y} = y$

(vi) $\dot{x} = x - y + xy, \quad \dot{y} = x - y - y^2$

Exercise 3.17 E

Show that the centres found by linearising the system in Exercise 3.16(ii) are centres of the non-linear system.

Exercise 3.18 S

Sketch the phase portraits in the neighbourhood of each of the fixed points of the system

$$\dot{x} = x(1 - x^2), \qquad \dot{y} = y,$$

and use your results to suggest a global phase portrait.

Exercise 3.19 S

Find and classify the fixed points of the following non-linear systems.

(i) $\ddot{x} + \dot{x} + x + x^3 = 0$

(ii) $\ddot{x} + \dot{x} + x - x^3 = 0$

Exercise 3.20 S

Classify the fixed point at the origin of the system

$$\dot{x} = -y + x(x^2 + y^2)^2, \qquad \dot{y} = x + y(x^2 + y^2)^2.$$

Exercise 3.21 E

Show that the following systems have no fixed points, and sketch their phase portraits.

(i) $\dot{x} = x + y + 2, \quad \dot{y} = x + y + 1$

(ii) $\dot{x} = e^{x+y}, \quad \dot{y} = x + y$

Exercise 3.22 S

Consider the second-order system with equation of motion $\ddot{x} = F(x, \dot{x})$, where $F(x, \dot{x})$ is an even function of $\dot{x}$. Show that if the system has fixed points then they are either centres or saddles.

Exercise 3.23 E

Show that, for the system

$$\dot{x} = x^2 - y^3, \qquad \dot{y} = x^2\left(x^2 - y^3\right),$$

there is a curve consisting of fixed points, and that every fixed point on this curve is not simple.

Is it always true for a system with a curve of fixed points, that every fixed point on this curve is not simple?

Exercise 3.24 E

A population of foxes feeds on a population of hares, the only source of food. Hares die as a result of encounters with foxes, so we take the proportionate death rate of the hares to be a decreasing linear function of the number of foxes. Their proportionate birth rate is assumed to be constant. The proportionate birth rate of the foxes is an increasing function of the amount of food available, that is of the number of hares; we take it to be a linear function. The proportionate death rate of the foxes is taken to be constant.

Show that with these assumptions, F, the number of foxes, and H, the number of hares, satisfy the equations

$$\frac{dH}{dt} = AH - BHF, \qquad \frac{dF}{dt} = CHF - DF,$$

where $F > 0$, $H > 0$ and A, B, C and D are positive constants.

Find the fixed point of the system, and describe the linearised motion in its neighbourhood. Interpret the qualitative features of this motion in terms of the populations of foxes and hares.

Exercise 3.25 E

For the population equations of the previous exercise, show that *all* phase curves are closed, and sketch a typical graph of $F(t)$.

If the population of hares is suddenly depleted by an epidemic disease from which the remaining hares are immune, discuss qualitatively the various different effects this could have on the population of foxes.

Exercise 3.26 S

A damped vertical pendulum has the equation of motion

$$\ddot{\psi} + \alpha\dot{\psi} + \omega^2 \sin\psi = 0, \qquad \alpha \geq 0,$$

where ψ is the angle between the pendulum and the downward vertical. Determine the position and the nature of the fixed points. Sketch the phase curves in a sufficiently large region of phase space to show all the qualitative features of the motion. Describe what happens to the motion and fixed points when $\alpha = 0$.

Exercise 3.27 E

Consider the autonomous system

$$\dot{x} = u(x,y), \qquad \dot{y} = v(x,y).$$

Show that the necessary condition for a function $f(x,y)$ to be constant along all phase curves of the system is that it should satisfy the partial differential equation

$$u(x,y)\frac{\partial f}{\partial x} + v(x,y)\frac{\partial f}{\partial y} = 0.$$

For the case

$$\dot{x} = xy, \qquad \dot{y} = \ln x,$$

show that the phase curves are given by

$$y^2 - (\ln x)^2 = \text{constant},$$

and hence deduce that a solution of the differential equation

$$xy\frac{\partial f}{\partial x} + \ln x\frac{\partial f}{\partial y} = 0$$

is $f(x,y) = y^2 - (\ln x)^2$.

[Note that $\displaystyle\int dx\,\frac{\ln x}{x} = \tfrac{1}{2}(\ln x)^2$.]

Exercise 3.28 E

In general, non-autonomous systems are difficult to treat. In particular, there is no simple classification scheme; but for linear systems with periodic coefficients, which can be written in the form

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}, \quad \mathbf{A}(t+T) = \mathbf{A}(t), \qquad \text{for all } t,$$

where $\mathbf{A}$ is a real, non-singular, T-periodic matrix, there is a very powerful theory. Let $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ be two linearly independent solutions of the equation of motion, such that $[\mathbf{x}_1(0) \quad \mathbf{x}_2(0)] = \mathbf{I}$ (the identity matrix); then any solution $\mathbf{z}(t)$ can be written as a linear combination of $\mathbf{x}_1$ and $\mathbf{x}_2$,

$$\mathbf{z}(t) = a\mathbf{x}_1(t) + b\mathbf{x}_2(t) = \begin{bmatrix} x_{11}(t) & x_{21}(t) \\ x_{12}(t) & x_{22}(t) \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \mathbf{X}(t)\mathbf{a},$$

for some constants a and b. This fact is a consequence of the linearity of the equations; for the purpose of this question, it is to be assumed. Note that $\mathbf{a} = \mathbf{z}(0)$.

Show that the following statements are true.

(i) If $\mathbf{x} = \mathbf{z}(t)$ is a solution of the equation $\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}$ then so is $\mathbf{x} = \mathbf{z}(t+T)$.

(ii) If $\mathbf{z}(t) = \mathbf{X}(t)\mathbf{a}$, where $\mathbf{X}$ is defined above, then

$$\mathbf{z}(nT) = \mathbf{X}(T)^n \mathbf{a}.$$

Let λ_1 and λ_2 be the eigenvalues of $\mathbf{X}(T)$, real or complex, distinct or coincident. Deduce that

- $\mathbf{z}(t)$ is unbounded as $t \to \infty$ if either $|\lambda_1| > 1$ or $|\lambda_2| > 1$, or both;
- $\mathbf{z}(t) \to \mathbf{0}$ as $t \to \infty$ if both $|\lambda_1| < 1$ and $|\lambda_2| < 1$;
- $\mathbf{z}(t)$ is oscillatory if $|\lambda_1| = |\lambda_2| = 1$.

Exercises on Unit 4

Exercise 4.1 S

Write down Hamilton's equations for a Hamiltonian function $H(q,p,t)$.

Give necessary and sufficient conditions for a second-order system

$$\dot{q} = Q(q,p,t), \qquad \dot{p} = P(q,p,t)$$

to be Hamiltonian with q and p as conjugate variables.

Exercise 4.2 S

Determine which of the following systems are Hamiltonian with x and y as conjugate variables, and for those that are, find a Hamiltonian function.

(i) $\dot{x} = xy, \quad \dot{y} = -xy + \sin(yt)$

(ii) $\dot{x} = x^2 y, \quad \dot{y} = x^3 - xy^2$

(iii) $\dot{x} = yt, \quad \dot{y} = xt^2$

(iv) $\dot{x} = -y^2, \quad \dot{y} = -x^2$

(v) $\dot{x} = x - 5y, \quad \dot{y} = x - y$

(vi) $\dot{x} = x + y, \quad \dot{y} = x - 2y$

Exercise 4.3 S

Show that the system

$$\dot{x} = y^2, \quad \dot{y} = -\frac{x}{2y}, \qquad y > 0,$$

is not Hamiltonian with respect to the given variables.

Define new variables $q = x$, $p = y^2$, and show that in this representation the system is Hamiltonian with Hamiltonian function

$$H(q,p) = \tfrac{1}{2}(p^2 + q^2).$$

Exercise 4.4 S

Let V be any well-behaved function of a single variable, and F a constant. Determine Newton's equations of motion for each of the three Hamiltonians

$$H_1(q_1,p_1,t) = \frac{1}{2m}p_1^2 + V(q_1) + Fq_1 \sin\omega t,$$

$$H_2(q_2,p_2,t) = \frac{1}{2m}p_2^2 + V(q_2) + \frac{Fp_2}{m\omega}\cos\omega t,$$

$$H_3(q_3,p_3,t) = \frac{1}{2m}p_3^2 + V\left(q_3 + \frac{F}{m\omega^2}\sin\omega t\right).$$

(In the last of these, the whole of the final expression in brackets is the argument of the function V.)

Show that Newton's equations for H_1 and H_2 are identical, and that Newton's equation for H_3 may be cast in the same form by using a suitable transformation.

Exercise 4.5 E

Consider a particle in the form of a bead, moving on a smooth wire in a vertical plane, whose shape is that of the graph of some function. The configuration of the system consisting of the bead on the wire may be specified by a single coordinate q along a horizontal axis, so that the position of any point of the wire has coordinates $(q, f(q))$, where $f(q)$ is the function giving the shape of the wire. The force of reaction between the bead and the wire is perpendicular to the wire, because the wire is assumed to be smooth. Suppose that the only other force acting is gravity.

(i) By taking components in the direction of the tangent to the wire, find an equation of motion for the bead which does not involve the reaction force.

(ii) Let

$$H(q,p) = \frac{p^2}{2m\sigma(q)^2} + mgf(q),$$

where $\sigma(q) = \sqrt{1 + f'(q)^2}$. Show that Hamilton's equations for this Hamiltonian are equivalent to the equation of motion you found in part (i).

Exercise 4.6 E

Write the second-order equation

$$\ddot{q} + G(q)\dot{q}^2 - F(q) = 0$$

as a pair of coupled first-order equations using the extra variable y with $y = \dot{q}$ in the usual way, and show that the system is not Hamiltonian with respect to the variables q and y.

Define the momentum p by

$$p = f(q)\dot{q},$$

where the function $f(q)$ is any solution of the differential equation

$$f'(q) = 2G(q)f(q)$$

(so that we can take

$$f(q) = \exp\left(2\int dq\, G(q)\right)).$$

Show that with respect to q and p the system is Hamiltonian, with Hamiltonian function

$$H(p,q) = \frac{p^2}{2f(q)} + V(q), \qquad \text{where} \quad V'(q) = -f(q)F(q).$$

Exercise 4.7 S

What does it mean to say that a Hamiltonian system is conservative?

Show that the time-dependent system with Hamiltonian

$$K(q,p,t) = f(t)H(q,p),$$

where $H(q,p)$ is independent of the time, is not conservative; but that it becomes conservative if the new time variable τ, defined by $\tau = \int dt\, f(t)$, is used instead of t.

Exercise 4.8 S

Show that, for a conservative Hamiltonian system, the square of the Hamiltonian function is conserved. Show, more generally, that if the motion of a system is determined by the Hamiltonian $H(q,p)$, and if a function $f(q,p)$ can be expressed as a differentiable function of H, so that $f(q,p) = F(H(q,p))$ where F is a function of a single variable, then $f(q,p)$ is constant along all phase curves.

Exercise 4.9 S

Show that the linear system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, where $\mathbf{A}$ is the real constant 2×2 matrix

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

is Hamiltonian if and only if $\operatorname{tr} \mathbf{A} = 0$, and that this implies that the fixed point at the origin is either a centre or a saddle.

Exercise 4.10 S

(i) What is a separatrix?

(ii) Sketch the phase curves of a particle of mass m moving in the potential

$$V(q) = -Aq^4, \qquad A > 0.$$

Give the equation of the separatrix, and find the motion, $q_s(t)$, on it. Show that this motion terminates.

(iii) The particle starts at time $t = 0$ at the point $q_0 > 0$ with momentum $p_0 > 0$. Obtain an expression for the time t at which it reaches the point $q > q_0$, in the form of an integral, and hence show that this motion also terminates.

Exercise 4.11 S

Sketch the potential function and the contours of the Hamiltonian

$$H(q,p) = \tfrac{1}{2}p^2 + \tfrac{1}{4}q^4 - \tfrac{1}{2}q^2.$$

Give the equation of the separatrix, and provide a qualitative description of the motion. Obtain the approximate period of small oscillations in the neighbourhood of any of the stable fixed points.

Exercise 4.12 S

Show that, for a system with Hamiltonian of mechanical type,

$$H(q,p) = \frac{p^2}{2m} + V(q),$$

the time T taken for a phase point to move from the configuration q_1 to the configuration $q_2 > q_1$ along a phase curve of energy E is given by

$$T = \sqrt{\frac{m}{2}} \int_{q_1}^{q_2} \frac{dq}{\sqrt{E - V(q)}}.$$

Use this result to find an integral expression for the period of oscillatory motion with energy E in such a system.

A particle of mass m has position $r > 0$ and moves under the action of the *force*

$$F(r) = -kr + \frac{\alpha}{r^3}, \qquad r > 0,$$

where k and α are positive constants.

Show that the system has a single fixed point, find its position, and determine the potential function $V(r)$ which is zero at the fixed point. Hence provide a qualitative description of the motion. Show that the fixed point is stable, and find the period of oscillation about it as a function of energy E for all E at which the motion is oscillatory.

Exercise 4.13 E

An electrically-charged particle of mass m is constrained to lie on the straight line between two fixed charges C_1 and C_2 at a distance $2l$ apart. The total force on the particle is the sum of the two forces of magnitude μ/r_j^2, where r_j is the distance from the particle to C_j, and each fixed charge repels the particle. Obtain the Hamiltonian and sketch the phase diagram.

The particle starts from rest at a distance kl from the mid-point of C_1 and C_2, where $0 < k < 1$. Show that it oscillates with period

$$T = 2\sqrt{\frac{ml^3(1-k^2)}{\mu}} \int_0^{\pi/2} d\theta \sqrt{1-k^2\sin^2\theta}.$$

Exercise 4.14 S

Show that for an even potential, $V(-q) = V(q)$, which supports oscillatory motion, the period of oscillations with energy E can be written in the form

$$T(E) = 2\sqrt{2m}\int_0^{q_2(E)} \frac{dq}{\sqrt{E-V(q)}}, \qquad \text{where} \quad V(q_2) = E.$$

Show that the period of motion in the potential

$$V(q) = \left(|q|/l\right)^\alpha, \qquad \alpha > 0,$$

is $$T(E) = \frac{2l\sqrt{2m}}{\alpha E^{(\alpha-2)/2\alpha}} J(\alpha),$$

where $$J(\alpha) = \int_0^1 \frac{dx}{x^{(1-1/\alpha)}\sqrt{1-x}}.$$

Exercise 4.15 E

(i) Sketch the phase curves for the motion in the even potential

$$V(q) = Aq(2a-q), \qquad q > 0,$$
$$V(-q) = V(q),$$

where A and a are positive constants.

(ii) Show that the period of librational motion can be expressed as a function of the energy, E, in the form

$$T(\varepsilon) = \sqrt{\frac{2m}{A}}\ln\left(\frac{1+\sqrt{\varepsilon}}{1-\sqrt{\varepsilon}}\right), \qquad \text{where} \quad \varepsilon = E/Aa^2.$$

(iii) Show that as the energy approaches the separatrix energy, the period diverges logarithmically (that is, in this limit, it behaves approximately like a logarithm function).

Exercise 4.16 S

Describe the phase space for rotational motion, that is, for a second-order system whose configuration space is a circle.

Explain the difference between a rotation and a libration, and give an example of each.

Exercise 4.17 E

Consider motion in a circle with the Hamiltonian

$$H(q,p) = \tfrac{1}{2}p^2 + p\sin q,$$

where we take $-\pi < q \le \pi$. Determine the position and nature of the fixed points. Sketch the phase diagram, showing clearly where there is libration and where there is rotation. Show that the period of the librational motion at energy E is

$$T(E) = 4\int_0^\alpha \frac{dx}{\sqrt{\cos^2 x - 2|E|}}, \qquad \text{where} \quad \cos^2\alpha = 2|E|.$$

Exercise 4.18 E

Find the fixed points and sketch the phase curves for the Hamiltonian

$$H(q,p) = \tfrac{1}{2}p^2 - F\cos nq, \qquad 0 \le q < 2\pi,$$

n being an integer.

Exercise 4.19 E

(i) Consider the rotational motion of a system with Hamiltonian

$$H(q,p) = \frac{1}{2m}p^2 + V(q), \qquad 0 \le q < \pi.$$

Show that the period T of the rotational motion can be written as a function of the energy in the form

$$T(E) = \sqrt{2m}\frac{d}{dE}\int_0^{2\pi} dq\sqrt{E - V(q)}.$$

(ii) Use this result to show that if

$$V(q) = V_0(q) + \varepsilon V_1(q),$$

then for sufficiently small ε the period can be written in the approximate form

$$T(E) = T_0(E) + T_1(E),$$

where $T_0(E)$ is the period for $V_0(q)$ alone and

$$T_1(E) = -\varepsilon\sqrt{\frac{m}{2}}\frac{d}{dE}\int_0^{2\pi} dq\frac{V_1(q)}{\sqrt{E - V_0(q)}}$$

$$= -\varepsilon\frac{d}{dE}\int_0^{T_0} dt\, V_1(q_0(t)),$$

where $q_0(t)$ is the motion in the potential $V_0(q)$.

Exercise 4.20 E

A vertical pendulum comprises a mass m attached to a rod of length l which swings freely in the range $-\alpha < q < \pi$ for some constant positive angle α, but is excluded from the range $-\pi < q < -\alpha$ by barriers at $q = -\alpha$ and π, as shown.

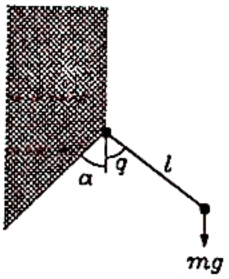


Figure 4.1

At the barriers the pendulum bounces elastically, that is, the momentum is instantaneously reversed. Sketch the phase curves of the system, and write down integral expressions for the period of the motion in all possible cases.

Exercise 4.21 E

Find and classify the fixed points, sketch the phase diagram, and describe the motion for the Hamiltonian

$$H(q,p) = p + \sqrt{1-p^2}\sin q, \qquad |p| \le 1.$$

Solutions to End of Block Exercises

Solution 2.1

Any set of specific values of the variables which correspond to the most significant features of a system defines a state of the system. The variables which define the states of a system may conveniently be thought of as the coordinates in a space: if there is one variable then this space may be a line, if there are two variables it may be a plane, and so on. Other spaces which may be given coordinates, such as the circle, are also permitted. The space of the state variables is called the phase space of the system. There is one essential criterion in choosing the state variables: it must be possible, in principle, to determine all future and all past states of the system if its present state is known.

States of a system, phase space and order are defined in Section 2.1.

The order of a system is the minimum number of coordinates required to specify a point of its phase space, in other words the dimension of the phase space.

(i) The phase space is the quadrant of the (p, q)-plane for which $p \geq 0$ and $q \geq 0$ (since population sizes cannot be negative). Thus the phase space has natural boundaries along the lines $p = 0$, $q \geq 0$ and $q = 0$, $p \geq 0$. This is a second-order system.

Natural boundaries are discussed in Subsection 2.4.2.

(ii) The plane is not a suitable phase space for this system because it does not satisfy the criterion of determinism: it is not sufficient to specify the particle's position alone in order to determine its past and future motion – despite the simplicity of the system. One needs to know the particle's velocity as well as its position to determine its motion, so the system is fourth-order.

Solution 2.2

(i) Second-order autonomous.

(ii) As the system is presented it is first-order non-autonomous (the variable x here plays the role of 'time'). However, if x is regarded as a dependent variable, the system may be treated as a second-order non-autonomous one with the equations of motion

$$\dot{u} = x^2 + u^2, \qquad \dot{x} = 1.$$

Note, however, that this system has motions which do not correspond to any possible motions of the original one, namely those for which $x = t + k$ with k any non-zero constant.

(iii) On putting $w = dy/dx$ this second-order equation becomes equivalent to the pair of coupled first-order equations

$$\frac{dy}{dx} = w, \qquad \frac{dw}{dx} = y^2.$$

So the system is second-order, with x in the role of 'time'. As the independent variable x does not occur on the right-hand sides of the above equations, the system is autonomous.

(iv) Third-order autonomous; the system is equivalent to

$$\dot{q} = p, \quad \dot{p} = r, \quad \dot{r} = -q.$$

(v) Second-order non-autonomous, unless $a = b = 0$, in which case it is autonomous.

The equations of motion of a system are discussed initially in Section 2.1. The reduction of second-order differential equations to systems of first-order ones is described in several places, including *Unit 3* Section 3.1, where the trick for making non-autonomous first-order systems into autonomous second-order ones is also treated (Exercise 3.5).

Solution 2.3

The equations of motion are

$$\dot{x} = x, \qquad \dot{y} = y.$$

The velocity field of a system is defined in Section 2.1.

Solution 2.4

For the system $\dot{x} = v(x)$ the time taken in going from $x = A$ to $x = B$ is

$$T = \int_A^B \frac{dx}{v(x)}.$$

Equation 2.3

Since $v_i(x) > 0$, $i = 1, 2$, we have $1/v_1(x) \geq 1/v_2(x)$, so that

$$\int_A^B \frac{dx}{v_1(x)} \geq \int_A^B \frac{dx}{v_2(x)},$$

and hence

$$T_2 \leq T_1.$$

Solution 2.5

Draw at each point on the x-axis an arrow of length proportional to $|v(x)|$ (where x is the point's coordinate and $v(x)$ is the phase velocity there), based on the point and pointing in the positive direction if $v(x) > 0$, the negative direction if $v(x) < 0$. At points where $v(x) = 0$ one draws not an arrow, but simply a spot. The corresponding collection of arrows is a picture of the velocity field of the system. For a first-order autonomous system one may easily draw on the same diagram the graph of $v(x)$ and use it to read off the information required to construct the arrows.

Graphical methods for autonomous first-order systems are discussed in Subsection 2.3.1.

In principle, the phase diagram contains all the information required to give a qualitative account of the system's motion, without integrating the equation of motion (which it may not be possible to do analytically anyway). Of particular significance are the fixed points of the system, where $v(x) = 0$; the most important use of the phase diagram is to locate the fixed points and determine the direction of motion in the intervals between them, since these are the main qualitative features of the motion.

Solution 2.6

(i) A fixed point of an autonomous first-order system is a zero of its velocity field, in other words a root of the equation $v(x) = 0$. A system initially at a fixed point remains there for all time. Fixed points are also called equilibrium points.

Fixed points and invariant sets of autonomous first-order systems are defined in Subsection 2.3.1. Stability and related ideas are discussed in the following subsection.

An invariant set of states is defined by the property that if the system is in such a set at some time it remains in that set for all time (the set in question is assumed to be non-empty). In other words, if ϕ_t is the flow of the system, a set S is invariant if $\phi_t(S) \subset S$ for all t. Any collection of fixed points is an invariant set, and so is the interval between any pair of fixed points. An invariant set is said to be elementary if it cannot be decomposed into smaller invariant sets. A single fixed point constitutes an elementary invariant set, and so does the interval between adjacent isolated fixed points.

The flow of a system is defined in Subsection 2.3.1.

The fixed points for $v(x) = x(1 - x)$ are $x = 0$ and $x = 1$, and the elementary invariant sets are these fixed points and the intervals $(-\infty, 0)$, $(0, 1)$ and $(1, \infty)$.

(ii) A simple fixed point is one at which the derivative of the velocity function does not vanish; it is thus a point x_f such that

$$v(x_f) = 0 \qquad \text{but} \qquad v'(x_f) \neq 0.$$

In the example, both $x = 0$ and $x = 1$ are simple fixed points.

(iii) A fixed point x_f such that $v'(x_f) < 0$, so that $v(x)$ is a decreasing function near x_f and neighbouring states approach x_f as t increases, is stable. A fixed point x_f such that $v'(x_f) > 0$, so that $v(x)$ is an increasing function near x_f and neighbouring states leave x_f as t increases, is unstable.

In the example, the fixed point at $x = 0$ is unstable, while that at $x = 1$ is stable.

(iv)

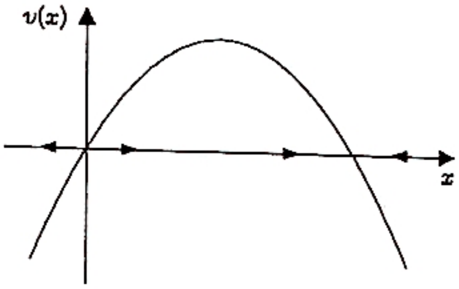


Figure S2.1 Graph of the velocity function $v(x) = x(1-x)$.

Solution 2.7

The mass M of a spherical drop is equal to the product of its volume and its density ρ,

$$M = \tfrac{4}{3}\pi R^3 \rho,$$

R being the drop's radius. Differentiation with respect to time gives

$$\frac{dM}{dt} = 4\pi R^2 \rho \dot{R} = S\rho \dot{R},$$

S being the surface area. But we are given that $\dot{M} = -kS$, for some positive constant k, so that

$$\frac{dR}{dt} = -\frac{k}{\rho}.$$

Thus $R(t)$ decreases at a constant rate; explicitly, $R(t) = R_0 - kt/\rho$. Clearly when $R(t)$ becomes very small this model breaks down.

This is an example of a system with a uniform velocity field; such systems are discussed in Example 2.4.

Solution 2.8

(i) The graph of the velocity function $v(x) = x \sin x$ is shown in Figure S2.2.

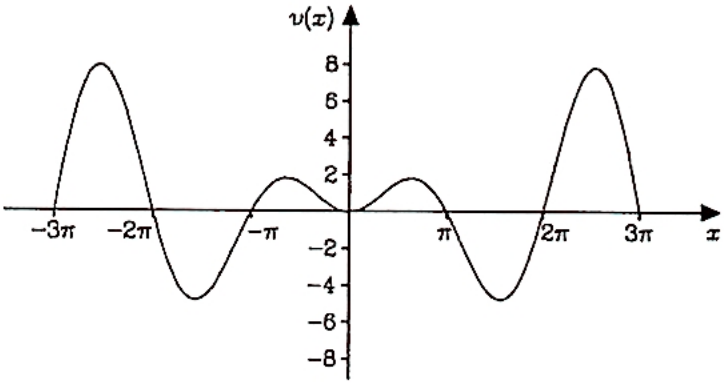


Figure S2.2 Graph of the velocity function $v(x) = x \sin x$.

The fixed points are at $x = n\pi$ for every integer n. Since

$$v'(x) = x \cos x + \sin x,$$

we have

$$v'(0) = 0 \qquad \text{and} \qquad v'(n\pi) = (-1)^n n\pi.$$

Thus for $n \neq 0$ the fixed points are simple; for n positive the fixed point at $n\pi$ is stable for n odd, unstable for n even; for n negative these conclusions are reversed. At the origin $v'(0) = 0$ and $v''(0) = 2$; this type of fixed point is discussed in Exercise 2.20 and is shown to be neither stable nor unstable, but to be like the fixed point at the origin of the system with velocity function x^2; Figure S2.2 confirms this.

The elementary invariant sets are the fixed points and the intervals $(n\pi, (n+1)\pi)$ for $n = \ldots, -2, -1, 0, 1, 2, \ldots$. For n odd and negative, for $n = 0$, and for n even and positive, the motion is to the right on $(n\pi, (n+1)\pi)$; in all other cases it is to the left.

(ii) The graph of the velocity function $v(x) = x\cos x$ is shown in Figure S2.3.

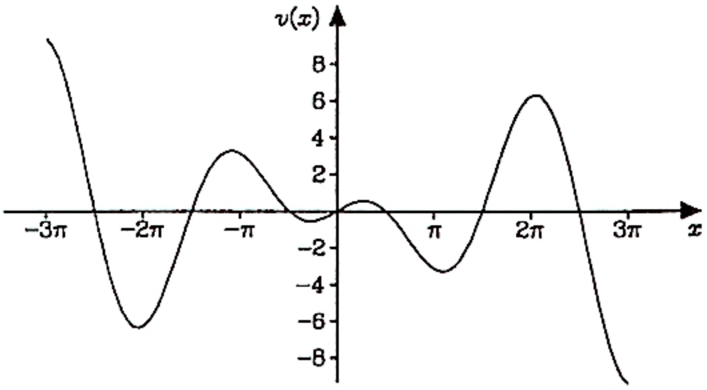


Figure S2.3 Graph of the velocity function $v(x) = x\cos x$.

The fixed points are at $x = 0$, $x = (n + \frac{1}{2})\pi$ for integers n. Since

$$v'(x) = \cos x - x\sin x,$$

we have

$$v'(0) = 1 \qquad \text{and} \qquad v'((n+\tfrac{1}{2})\pi) = -(-1)^n(n+\tfrac{1}{2})\pi.$$

All the fixed points are simple, and they are alternately stable and unstable with the origin being unstable. That is, the fixed points

$$x_f = (n+\tfrac{1}{2})\pi, \quad n = \ldots, -3, -1, 0, 2, 4, \ldots, \quad \text{are stable}$$

and

$$x_f = 0 \text{ and } x_f = (n+\tfrac{1}{2})\pi, \quad n = \ldots, -4, -2, 1, 3, \ldots, \quad \text{are unstable.}$$

The elementary invariant sets are the fixed points and the intervals $(-\frac{1}{2}\pi, 0)$, $(0, \frac{1}{2}\pi)$, and $((n-\frac{1}{2})\pi, (n+\frac{1}{2})\pi)$ for $n = \pm1, \pm2, \ldots$. The motion is to the right on the interval $(0, \frac{1}{2}\pi)$ and the intervals $((n-\frac{1}{2})\pi, (n+\frac{1}{2})\pi)$ with n odd and positive and n even and negative; it is to the left otherwise.

Solution 2.9

The flow $\phi_t(x)$ is the state of the system at time t if it is initially in the state x. As all the examples are autonomous we can take the initial time to be zero.

(i) Let $y = \phi_t(x)$; then the solution of the equation of motion can be written in terms of an integral as

$$\int_x^y \frac{dx}{x} = bt \qquad \text{or} \qquad \ln\left(\frac{y}{x}\right) = bt.$$

Hence

$$y = \phi_t(x) = xe^{bt}.$$

(ii) With notation as in part (i) we have

$$\int_x^y \frac{dx}{x^2} = bt \qquad \text{or} \qquad \frac{1}{x} - \frac{1}{y} = bt.$$

On rearranging this last equation we obtain

$$y = \phi_t(x) = \frac{x}{1 - xbt}, \qquad t < \frac{1}{bx}.$$

Notice that in this case $\phi_t(x)$ is defined only for a finite time, if $x \neq 0$, as the motion terminates at $t = 1/bx$.

(iii) In this case separation of variables gives

$$t = \int_x^y dx\,\frac{1}{x(1-x)} = \int_x^y dx\left(\frac{1}{x}+\frac{1}{1-x}\right)$$
$$= \left[\ln\left(\frac{x}{1-x}\right)\right]_x^y = \ln\left[\left(\frac{y}{1-y}\right)\left(\frac{1-x}{x}\right)\right].$$

This relation may be rearranged to give

$$\frac{y}{1-y} = \frac{x}{1-x}e^t$$

or $$y = \phi_t(x) = \frac{xe^t}{1-x+xe^t}.$$

(iv) Separation of variables gives

$$t = \int_x^y dx\,\frac{1}{x(1-x^2)} = \int_x^y dx\left(\frac{1}{x}+\frac{x}{1-x^2}\right)$$
$$= \left[\ln x - \tfrac{1}{2}\ln(1-x^2)\right]_x^y = \left[\tfrac{1}{2}\ln\left(\frac{x^2}{1-x^2}\right)\right]_x^y.$$

This is the same equation as in part (iii) with x replaced by x^2 and t by $2t$, so that

$$y = \phi_t(x) = \frac{xe^t}{\sqrt{1-x^2+x^2e^{2t}}}.$$

Solution 2.10

There are only two types of system, as the following argument shows. In general, adjacent simple fixed points must be of different type, as the phase velocity cannot change direction (sign) without passing through zero, and this would contradict the assumption that the two fixed points were adjacent. Thus the whole phase portrait is defined by, say, the leftmost fixed point. If this is stable the phase portrait looks like this

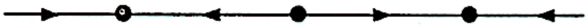


Figure S2.4

and if it is unstable it looks like this

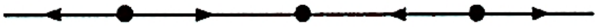


Figure S2.5

If the fixed points are not necessarily simple then there is no restriction on the directions of the arrows in adjacent intervals, and the phase portrait consists of four intervals (two finite, two infinite) with an arrow in each, which can point in either direction. There are thus sixteen possible essentially different phase portraits.

Solution 2.11

The fixed points are the roots of

$$x^3 + \varepsilon x = 0,$$

so $x = 0$ is always a fixed point, and it is the only one if $\varepsilon \geq 0$. If $\varepsilon < 0$, however, there are two additional fixed points, at $x = \pm\sqrt{-\varepsilon}$. Since

$$\overline{v}'(x) = 3x^2 + \varepsilon,$$

the fixed point at $x = 0$ is simple except when $\varepsilon = 0$; it is stable for $\varepsilon < 0$, unstable for $\varepsilon \geq 0$, the case $\varepsilon = 0$ being dealt with in Exercise 2.17. When $\varepsilon < 0$ the fixed points at $\pm\sqrt{-\varepsilon}$ are both simple and unstable.

These results may also be obtained readily from a consideration of the graph of $x^3 + \varepsilon x$.

Thus if one imagines ε decreasing from some initial positive value, the number and nature of the fixed points changes abruptly as it passes through the critical value 0. Before that value is reached there is a single unstable fixed point at the origin; as ε passes through 0 this fixed point changes type, and two new ones are born, both unstable. The fixed points are all simple except for the case when ε takes the critical value 0.

Solution 2.12

The simplest example of a structurally stable system is one with constant non-zero velocity function, $v(x) = a$. A system $\dot{x} = a + \varepsilon w(x)$ for which the perturbing function $\varepsilon w(x)$ satisfies the inequality $|\varepsilon w(x)| < |a|$ for all x cannot have a fixed point, and therefore has the same structure as the unperturbed system. Thus the flow of $\dot{x} = a$ retains its structure under all sufficiently small perturbations.

The effects of perturbations on the structure of the flow of a system are discussed in Subsection 2.3.3. The results of Exercises 2.24 and 2.23 are particularly relevant.

Every linear system $\dot{x} = a + bx$ with $b \neq 0$ is structurally stable, and in fact any system whose fixed points are all simple is structurally stable.

The system with the zero velocity function is clearly structurally unstable: every point is a fixed point of it, and yet (for example) by adding an arbitrarily small constant perturbing function one eliminates all its fixed points.

The system with velocity function $v(x) = x^2$ is another example of a structurally unstable one. This system has a single fixed point at $x = 0$, which is not simple. The perturbed system with velocity function

$$v(x) = x^2 + \varepsilon$$

has 2, 1 or 0 fixed points according as $\varepsilon < 0$, $\varepsilon = 0$ or $\varepsilon > 0$. Thus the structure of the flow of the system with $v(x) = x^2$ is radically altered by arbitrarily small perturbations of its velocity function. The system $\dot{x} = x^3$ considered in the previous exercise is yet another example.

It is worth emphasising that the notion of structural stability is quite different from that of the stability of a state. The stability of a state is concerned with an individual fixed point of a single given dynamical system, and is a matter of the behaviour of nearby phase points under that dynamical system. It involves no comparison with other systems. Structural stability, on the other hand, is concerned with the structure of the whole flow of a system, and its behaviour in relation to other systems.

Solution 2.13

Solving the equation of motion by separation of variables, we have

Terminating motion is dealt with in Subsection 2.4.1.

$$\int_{x_0}^{x} dx\, x^{-\alpha} = t,$$

and integration gives

$$\frac{1}{1-\alpha}\left(x^{1-\alpha} - x_0^{1-\alpha}\right) = t,$$

given that $\alpha \neq 1$, where $x_0 > 0$ is the initial state (at $t = 0$). It is clearly convenient to set $\alpha = 1 + \varepsilon$, $\varepsilon > 0$, whence

$$x(t) = \left(x_0^{-\varepsilon} - \varepsilon t\right)^{-1/\varepsilon},$$

and the motion terminates at $t = x_0^{-\varepsilon}/\varepsilon$.

This result generalises that of Exercise 2.27.

Suppose that $v(x) > kx^{1+\varepsilon}$ for all $x \geq A$, and that B is any number greater than A. Then by Exercise 2.4 above, the time taken by the system with velocity function $v(x)$ in going from state $x = A$ to state $x = B$ is less than that taken by the system with velocity function $kx^{1+\varepsilon}$; and this is true for any $B > A$, however large B may be. By a simple extension of the result of part (i), the second system terminates.

It follows that the first system must also terminate, and indeed in as short, if not a shorter time. Thus, for the system with velocity function $v(x)$, any motion for which the initial state $x_0 = A$ is sufficiently large as to lie in the range for which the inequality holds, terminates.

Solution 2.14

Systems of this kind are discussed in Subsection 2.4.2.

(i) The velocity function $v(x) = a\sqrt{x-b}$ is not real for $x < b$, so here the motion is not defined. For $x \geq b$, $v(x)$ is positive and monotonically increasing, as shown in Figure S2.6; so if $x(0) > b$ the phase point moves to the right with ever-increasing speed.

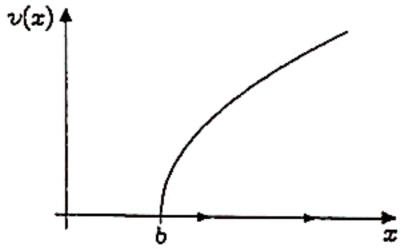


Figure S2.6 Graph of the velocity function $v(x) = a\sqrt{x-b}$, $a > 0$.

The only fixed point is $x = b$. At $x = b$, $v(b) = 0$, but $v'(b)$ and all higher derivatives are undefined, and in fact $v'(x)$ is unbounded as $x \to b$ from above, so even though $v(b) = 0$ the solution does not behave like that of a system with a simple zero. We can see this directly as this equation can be integrated quite easily. The system is separable, but it is easier to define $y^2 = x - b$, $x \geq b$, so that $\dot{x} = 2y\dot{y}$ and the equation of motion becomes

$$2\dot{y} = a$$

with solution

$$y(t) = y_0 + \tfrac{1}{2}at, \qquad y_0 > 0.$$

Note that this solution is valid only for $y_0 > 0$ as the derivation of the equation of motion involved a division by y. The equation for $x(t)$ is

$$\begin{aligned} x(t) &= b + \left(y_0 + \tfrac{1}{2}at\right)^2, \qquad \text{where} \quad y_0 = \sqrt{x_0 - b}, \\ &= x_0 + at\sqrt{x_0 - b} + \tfrac{1}{4}a^2t^2, \end{aligned}$$

which for small t increases linearly with t; quite different behaviour from a simple fixed point.

(ii) The motion is undefined for $x > b$, while for $x < b$ the solution of the equation of motion can be obtained by the method used in part (i) with $y^2 = b - x$, and is

$$y(t) = y_0 - \tfrac{1}{2}at, \qquad y_0 > 0,$$

or in terms of x,

$$\begin{aligned} x(t) &= b - \left(y_0 - \tfrac{1}{2}at\right)^2, \qquad \text{where} \quad y_0 = \sqrt{b - x_0}, \quad x_0 < b, \\ &= x_0 + (a\sqrt{b - x_0})t - \tfrac{1}{4}a^2t^2. \end{aligned}$$

The motion terminates at time $t = 2y_0/a$.

Solution 2.15

(i) In the vicinity of $x = b$, $v(x) \simeq a\sqrt{b - x}$, so the motion is undefined for $x > b$, while for $x < b$ the solution of the equation of motion is given approximately by the solution to Exercise 2.14(ii), and therefore terminates.

(ii) An example of a polynomial $g(x)$ with $N = 7$ zeros is shown in Figure S2.7.

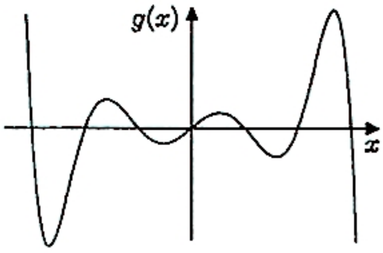


Figure S2.7 Graph of a polynomial with 7 zeros.

Between two consecutive zeros, $g(x)$ is either positive or negative. Where $g(x)$ is negative no motion of the system $\dot{x} = \sqrt{g(x)}$ is possible; but where $g(x)$ is positive $\dot{x} > 0$, so a phase point initially in one of these positive regions moves in the direction of increasing x, towards the larger zero. Near here the velocity function takes the form discussed in part (i), so the motion terminates at this zero.

Let the zeros of $g(x)$ be $x_1 < x_2 < x_3 < \cdots < x_N$. Then for $x < x_1$ there is no motion if $g(x) < 0$, but if $g(x) > 0$ the motion terminates at $x = x_1$. For $x > x_N$ there is no motion if $g(x) < 0$, but if $g(x) > 0$ then $\dot{x} > 0$ for all $x > x_N$ and $x(t)$ increases without bound. In fact, in this region $g(x) \sim x^N$ so $x(t)$ reaches infinity after a finite time.

Solution 2.16

The velocity functions in the two cases $\alpha < 2\beta$ and $\alpha > 2\beta$ are shown in the figure.

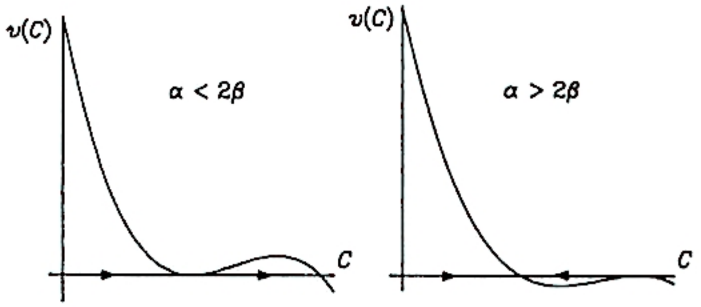


Figure S2.8 Graph of the velocity function $v(C) = k(\alpha - C)^2(2\beta - C)$, with the case $\alpha < 2\beta$ on the left and $\alpha > 2\beta$ on the right.

In both cases $v(0) > 0$ so $C(t)$ initially increases from zero, and continues to increase towards the first fixed point. When $\alpha < 2\beta$ this is at $C = \alpha$ and it is not a simple zero; near $C = \alpha$, $v(C)$ is approximately of quadratic form, so $C(t)$ reaches the value α after an infinite time. When $\alpha > 2\beta$ the fixed point at $C = 2\beta$ is a simple zero so $C(t)$ approaches 2β asymptotically, never actually reaching the fixed point in a finite time.

This example provides a clear indication of the power of geometric methods; merely by sketching the velocity functions we have been able to deduce that the behaviour of $C(t)$ is qualitatively different in the two cases.

In this particular instance the differential equation can be integrated by separating variables, as follows. We have

$$kt = \int_0^C \frac{dC}{(\alpha - C)^2(2\beta - C)};$$

then by expressing the integrand as a partial fraction, we obtain

$$kt = \int_0^C dC \left(\frac{A_1}{\alpha - C} + \frac{A_2}{(\alpha - C)^2} - \frac{A_1}{2\beta - C} \right),$$

where

$$A_1 = -\frac{1}{(\alpha - 2\beta)^2}, \qquad A_2 = \frac{1}{2\beta - \alpha}.$$

This integral then gives

$$kt = A_1 \ln \left(\frac{\alpha - C}{2\beta - C} \frac{2\beta}{\alpha} \right) + A_2 \left(\frac{1}{\alpha - C} - \frac{1}{\alpha} \right).$$

The simple behaviour deduced from the geometric description is not readily apparent from this formula, which we give only to reinforce the message.

Solution 2.17

The graph of the velocity function is sketched in the figure.

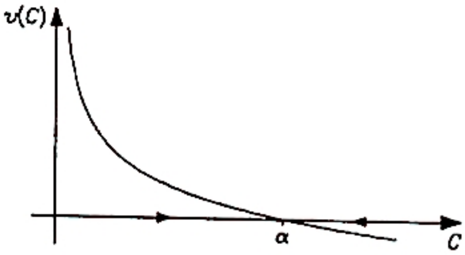


Figure S2.9 Graph of the velocity function $v(C) = k(\alpha - C)C^{-1/2}$.

This function is unbounded at $C = 0$ and has a simple zero at $C = \alpha$, and $v(C) > 0$ for $0 < C < \alpha$, so that the concentration $C(t)$ is an increasing function of time, asymptotically approaching the value α.

For $C \ll \alpha$ the equation of motion is approximately

$$\frac{dC}{dt} = \frac{k\alpha}{\sqrt{C}}, \qquad C(0) = 0,$$

which can be solved by separating the variables, that is,

$$\int_0^C dC \sqrt{C} = k\alpha t, \qquad \text{giving} \quad C(t) = \left(\frac{3k\alpha t}{2} \right)^{2/3},$$

so that $C(t)$ initially increases as $t^{2/3}$.

For $C(t)$ near α the equation of motion is approximately

$$\frac{dC}{dt} = \frac{k}{\sqrt{\alpha}}(\alpha - C),$$

which is linear and has a solution of the form

$$C(t) = \alpha - A \exp\left(-kt/\sqrt{\alpha}\right)$$

for some constant A: thus $C(t)$ approaches α as $t \to \infty$.

In this case the solution of the differential equation can be obtained in parametric form; that is, we can obtain $C(\theta)$ and $t(\theta)$ as functions of a parameter θ, rather than obtaining the solution as an explicit function $C(t)$. First, note that a rescaling of both variables, $C = \alpha x$ and $t = (\sqrt{\alpha}/k)\tau$, gives the simpler equation

$$\frac{dx}{d\tau} = \frac{1 - x}{\sqrt{x}},$$

which contains no unknown constants. Note that for $0 \leq C \leq \alpha$, we have $0 \leq x \leq 1$. The equation is separable and its solution can be written as the integral

$$\tau(x) = \int_0^x dz \, \frac{\sqrt{z}}{1 - z}.$$

Now for $0 \leq x \leq 1$ we can make the substitution $z = \sin^2 y$, to give

$$\tau = 2 \int_0^\theta dy \, \frac{\sin^2 y}{\cos y} = 2 \int_0^\theta dy \left(\frac{1}{\cos y} - \cos y \right),$$

where $\sin^2 \theta = x$. This equation can be integrated directly to give

$$\tau(\theta) = 2 \left[\ln(\tan(\tfrac{1}{2}\theta + \tfrac{1}{4}\pi)) - \sin \theta \right], \quad x(\theta) = \sin^2 \theta, \quad 0 \leq \theta < \pi/2.$$

By considering the approximations $\theta \simeq 0$ and $\theta \simeq \frac{1}{2}\pi$, the above limiting forms may be obtained; these provide a useful check on the more general analysis.

For small t, $C(t)$ increases fairly rapidly, as $t^{2/3}$, but as $C(t)$ approaches α its rate of increase becomes slower and slower, as shown in Figure S2.10.

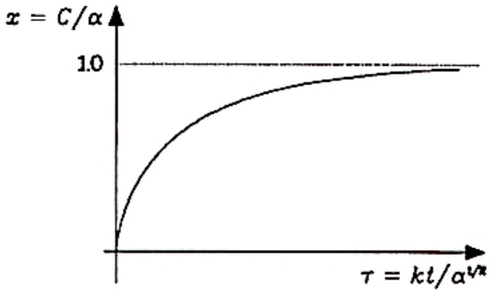


Figure S2.10 Graph of the motion produced by the velocity function $v(C) = k(\alpha - C)C^{-1/2}$.

Solution 2.18

Circular motion is a rotation if there is no fixed point, that is, if the equation $v(\theta) = 0$ has no solutions. In this case the relevant equation simplifies to

Rotational motion is discussed in Subsection 2.4.3.

$$k + \cos \theta = 0,$$

which has real solutions for $-1 \leq k \leq 1$ but none for $k > 1$ or $k < -1$; since the latter case is excluded (because then no real motion would be possible), the condition for the motion to be a rotation is that $k > 1$. The period is

$$\int_0^{2\pi} \frac{d\theta}{\sqrt{k + \cos \theta}}.$$

Solution 2.19

The velocity function $v(V)$ is sketched in Figure S2.11.

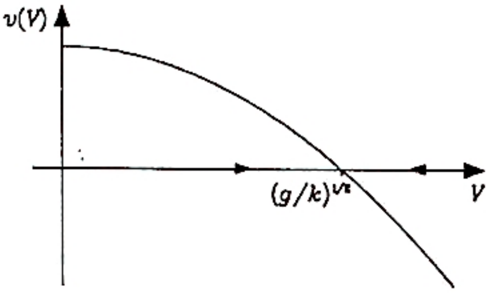


Figure S2.11 Graph of the velocity function $v(V) = g - kV^2$.

It has one simple fixed point at $V = \sqrt{g/k} = V_{\mathrm{T}}$, the terminal velocity. This fixed point is stable. For initial speeds $V(0) < V_{\mathrm{T}}$, $V(t)$ approaches V_{T} from below as $t \to \infty$, and for $V(0) > V_{\mathrm{T}}$, $V(t)$ approaches V_{T} from above as $t \to \infty$.

This is a case in which an explicit expression can be found for $V(t)$. On separating the variables the differential equation becomes

$$\int_{V_0}^{V} \frac{dV}{(g/k) - V^2} = kt.$$

The integral is easiest to evaluate by writing ${V_\mathrm{T}}^2$ for g/k and using partial fractions to put it in the form

$$kt = \frac{1}{2V_\mathrm{T}}\int_{V_0}^{V} dV\,\left(\frac{1}{V_\mathrm{T}-V}+\frac{1}{V_\mathrm{T}+V}\right),$$

which can be integrated to give

$$\frac{2gt}{V_\mathrm{T}} = \left[\ln\left(\frac{V_\mathrm{T}+V}{V_\mathrm{T}-V}\right)\right]_{V_0}^{V}.$$

This can be rewritten in the form

$$\frac{V_\mathrm{T}+V}{V_\mathrm{T}-V} = \left(\frac{V_\mathrm{T}+V_0}{V_\mathrm{T}-V_0}\right)\exp(2gt/V_\mathrm{T}),$$

which can be rearranged to give

$$V(t) = V_\mathrm{T}\left(\frac{(V_\mathrm{T}+V_0)\exp(2gt/V_\mathrm{T})-(V_\mathrm{T}-V_0)}{(V_\mathrm{T}+V_0)\exp(2gt/V_\mathrm{T})+(V_\mathrm{T}-V_0)}\right).$$

Solution 2.20

On substitution of tu for x the equation of motion becomes

$$\frac{dx}{dt} = t\frac{du}{dt}+u = v(ut,t) = v(u,1)$$

or

$$t\frac{du}{dt} = v(u,1)-u.$$

This now suggests defining a new variable $d\tau = dt/t$ or $\tau = \ln t$, so that

$$\frac{du}{dt} = \frac{du}{d\tau}\frac{d\tau}{dt} \qquad \text{or} \qquad t\frac{du}{dt} = \frac{du}{d\tau},$$

and the equation of motion becomes

$$\frac{du}{d\tau} = v(u,1)-u.$$

At a fixed point u_f of the new system, $v(u_\mathrm{f},1) = u_\mathrm{f}$, but from the original definition $x = u_\mathrm{f}t$, so x is increasing uniformly. More generally, near the fixed point

$$v(u,1)-u = k(u-u_\mathrm{f}) + O\big((u-u_\mathrm{f})^2\big)$$

with $k \neq 0$, and

$$\frac{du}{d\tau} = k(u-u_\mathrm{f}) \qquad \text{or} \qquad u(\tau) = u_\mathrm{f} + Ae^{k\tau}$$

for some constant A. In the original (x,t)-representation this is

$$x(t) = t\left(u_\mathrm{f} + At^k\right).$$

For a stable fixed point, $k < 0$, so the second term tends to zero as $t \to \infty$, and so the motion of the original system is asymptotically uniform. For an unstable fixed point, $k > 0$, and the term t^k grows without bound.

The condition for asymptotically uniform motion is

$$\frac{\partial v}{\partial x}(u_\mathrm{f},1) < 1.$$

Solution 2.21

(i) For any sufficiently well-behaved function $v(t)$ the equation $\dot{x} = xv(t)$ can be integrated by separating the variables:

$$\int_{x(a)}^{x(b)}\frac{dx}{x} = \int_a^b dt\,v(t) \qquad \text{or} \qquad \ln\left(\frac{x(b)}{x(a)}\right) = \int_a^b dt\,v(t),$$

giving the required result

$$x(b) = x(a)\exp\left(\int_a^b dt\,v(t)\right). \tag{S2.1}$$

(ii) On putting $a = \tau$, $b = NT + \tau$, and splitting the range of integration $(\tau, NT + \tau)$ into the N ranges

$$((n-1)T + \tau, nT + \tau), \qquad n = 1, 2, \dots, N,$$

we can write the integral as a sum over N terms,

$$\int_a^b dt\, v(t) = \sum_{n=1}^{N} \int_{(n-1)T+\tau}^{nT+\tau} dt\, v(t).$$

But each term in this sum is an integral over a complete period of $v(t)$ so all terms have the same value.

This can be shown formally by changing variables. Consider the nth term: define $t = (n-1)T + s$, so this integral is

$$\int_{(n-1)T+\tau}^{nT+\tau} dt\, v(t) = \int_{\tau}^{T+\tau} ds\, v((n-1)T + s).$$

But $v(T + t) = v(t)$, so for any m, $v(mT + t) = v(t)$; in particular, on putting $m = n - 1$ we find that

$$\int_{(n-1)T+\tau}^{nT+\tau} dt\, v(t) = \int_{\tau}^{T+\tau} ds\, v(s).$$

The last stage is to note that when integrating over a complete period of a periodic function the starting point is irrelevant. Formally, we write

$$\int_{\tau}^{T+\tau} ds\, v(s) = \int_{\tau}^{T} ds\, v(s) + \int_{T}^{T+\tau} ds\, v(s),$$

and in the last integral put $s = T + t$ so it becomes

$$\int_{T}^{T+\tau} ds\, v(s) = \int_0^{\tau} dt\, v(T + t) = \int_0^{\tau} dt\, v(t).$$

It follows that

$$\int_{\tau}^{NT+\tau} dt\, v(t) = N \int_0^T dt\, v(t).$$

Returning now to the solution:

$$\begin{aligned} x(NT + \tau) &= x(\tau) \exp\left(\int_{\tau}^{NT+\tau} dt\, v(t) \right) \\ &= x(\tau) \exp\left(N \int_0^T dt\, v(t) \right) = \mu^N x(\tau), \end{aligned}$$

where

$$\mu = \exp\left(\int_0^T dt\, v(t) \right),$$

as required.

(iii) Since $v(t)$ is integrable in $(0, T)$, $x(t)$ is bounded in $(0, T)$. The value of $x(t)$ for $t > T$ can be derived from the result of part (ii). First note that $\mu > 0$.

If $\mu > 1$ then μ^N grows without bound as $N \to \infty$, and hence $x(t) \to \infty$ with t. If $\mu < 1$ then $\mu^N \to 0$ as $N \to \infty$ so that $x(t) \to 0$ as $t \to \infty$. If, however, $\mu = 1$ then $x(T + \tau) = x(\tau)$ so $x(t)$ is periodic, and is therefore bounded as $t \to \infty$.

(iv) Using the notation of the previous part we have

$$\mu = \exp\left(\int_0^T dt\, v(t) \right) = \exp\left(\int_0^{\alpha T} dt - \int_{\alpha T}^{T} dt \right) = \exp(T(2\alpha - 1)).$$

Thus the solution $x(t)$ is bounded if $2\alpha - 1 \leq 0$, that is, $\alpha \leq \frac{1}{2}$. For larger values of α, $x(t)$ increases exponentially.

Solution 3.1

(i) Define a new variable $y = t$; then $\dot{y} = 1$ and the equation becomes

$$\dot{x} = -1 - xy, \qquad \dot{y} = 1.$$

(ii) Define a new variable $y = \dot{x}$, to give the equations

$$\dot{x} = y, \qquad \dot{y} = x^3 - x.$$

(iii) Define a new variable $y = \dot{x}$, to give the equations

$$\dot{x} = y, \qquad \dot{y} = -4x^2 - 3y^2.$$

An example of this procedure is given in Exercise 3.5.

Conversion of second-order equations to second-order systems is treated in Subsection 3.2.1.

Solution 3.2

Since the equation is linear, the sum of any two solutions is a solution, and a constant multiple of any solution is a solution. Thus since $\phi_t(\mathbf{x})$ and $\phi_t(\mathbf{y})$ are both solutions of the equation (regarding t as the variable), so is $a\,\phi_t(\mathbf{x}) + b\,\phi_t(\mathbf{y})$. Moreover, since $a\,\phi_0(\mathbf{x}) + b\,\phi_0(\mathbf{y}) = a\mathbf{x} + b\mathbf{y}$, this solution must be identical with the one which takes the initial value $a\mathbf{x} + b\mathbf{y}$; and this is just $\phi_t(a\mathbf{x} + b\mathbf{y})$.

The flow of an autonomous second-order system is defined, and its properties are discussed, in Subsection 3.2.2.

Any vector-valued linear function of a vector argument may be represented by a matrix, and if we choose to call the matrix in this case $\hat{\mathbf{A}}(t)$ we may write $\phi_t(\mathbf{x}) = \hat{\mathbf{A}}(t)\mathbf{x}$. From the fundamental properties

Exercise 3.7

$$\phi_t(\phi_s(\mathbf{x})) = \phi_s(\phi_t(\mathbf{x})) = \phi_{s+t}(\mathbf{x}), \qquad \phi_0(\mathbf{x}) = \mathbf{x},$$

we immediately obtain

$$\hat{\mathbf{A}}(s)\hat{\mathbf{A}}(t)\mathbf{x} = \hat{\mathbf{A}}(s+t)\mathbf{x}, \qquad \hat{\mathbf{A}}(0)\mathbf{x} = \mathbf{x}.$$

Since these hold for all $\mathbf{x}$, it follows that

$$\hat{\mathbf{A}}(s)\hat{\mathbf{A}}(t) = \hat{\mathbf{A}}(s+t), \qquad \hat{\mathbf{A}}(0) = \mathbf{I}.$$

The vector function $\mathbf{x}(t) = \hat{\mathbf{A}}(t)\mathbf{x}$ is a solution of the differential equation $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, from which it follows that

$$\frac{d\hat{\mathbf{A}}}{dt}(t)\mathbf{x} = \mathbf{A}\hat{\mathbf{A}}(t)\mathbf{x},$$

and since (again) this holds for all $\mathbf{x}$,

$$\frac{d\hat{\mathbf{A}}}{dt}(t) = \mathbf{A}\hat{\mathbf{A}}(t).$$

By repeated differentiation we obtain

$$\frac{d^n\hat{\mathbf{A}}}{dt^n}(t) = \mathbf{A}^n\hat{\mathbf{A}}(t);$$

and since $\hat{\mathbf{A}}(0) = \mathbf{I}$,

$$\frac{d^n\hat{\mathbf{A}}}{dt^n}(0) = \mathbf{A}^n.$$

Thus the Taylor expansion of $\hat{\mathbf{A}}(t)$ is

$$\begin{aligned}\hat{\mathbf{A}}(t) &= \hat{\mathbf{A}}(0) + t\frac{d\hat{\mathbf{A}}}{dt}(0) + \frac{1}{2!}t^2\frac{d^2\hat{\mathbf{A}}}{dt^2}(0) + \frac{1}{3!}t^3\frac{d^3\hat{\mathbf{A}}}{dt^3}(0) + \cdots \\ &= \mathbf{I} + t\mathbf{A} + \frac{1}{2!}t^2\mathbf{A}^2 + \frac{1}{3!}t^3\mathbf{A}^3 + \cdots.\end{aligned}$$

(The matrix $\hat{\mathbf{A}}(1)$ is usually called the (matrix) exponential of $\mathbf{A}$, written $\exp(\mathbf{A})$, and $\hat{\mathbf{A}}(t) = \exp(t\mathbf{A})$.)

Solution 3.3

(i) The equations of motion can be integrated directly to give

$$x(t) = x_0 + at, \qquad y(t) = y_0 e^t.$$

Examples 3.1 and 3.2, and Exercises 3.11 and 3.12, consider similar velocity fields.

The phase curves are given by

$$y = y_0 \exp\left(\frac{x - x_0}{a}\right).$$

This expression can be obtained either by eliminating t from the formulae for $x(t)$ and $y(t)$, or by noting that the phase curves satisfy the equation

$$\frac{dy}{dx} = \frac{y}{a}.$$

The system has no fixed points; $x(t)$ is increasing with t, while $y(t)$ increases if $y_0 > 0$ and decreases if $y_0 < 0$. The phase curves are shown in Figure S3.1; note that the x-axis is a phase curve.

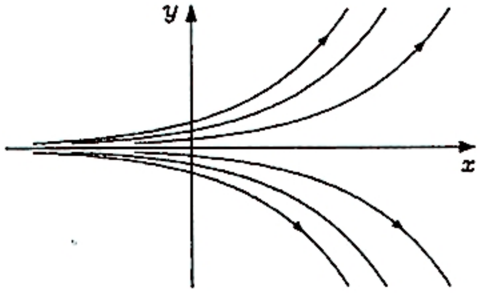


Figure S3.1

(ii) If we put $X = y$ and $Y = x$ we find that $(\dot{X}, \dot{Y}) = (a, Y)$, so the system is essentially the same as that treated in part (i): in fact, its phase curves can be obtained from those shown in the figure above by reflection in the line $y = x$.

(iii) The phase curves are solutions of the equation

$$\frac{dy}{dx} = \frac{3x^2}{2y}$$

and are therefore given by

$$y^2 - y_0^2 = x^3 - x_0^3.$$

From the original equations of motion we see that the phase curves are parallel to the x-axis when $x = 0$ and perpendicular to this axis when $y = 0$. Moreover, the phase curves for $y > 0$ and $y < 0$ are obtained from each other by a reflection in the x-axis (and a time-reversal). It is clear from the equation for the phase curves that every phase curve crosses the x-axis, and does so just once. This means that in order to examine the phase curves it is enough to consider those for which $y_0 = 0$. Then on each phase curve $x \geq x_0$. For large positive x we have $y \simeq \pm x^{3/2}$. The phase curve through the origin, for which $x_0 = 0$, is given by $y^2 = x^3$; it is defined only for $x > 0$, is symmetric about the x-axis and has zero gradient at the origin.

The phase curves are sketched in Figure S3.2.

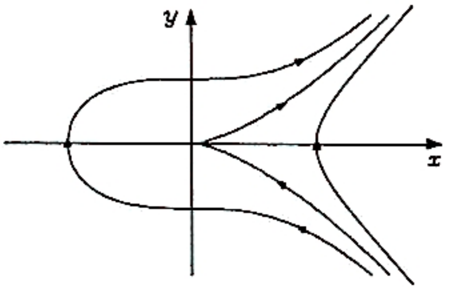


Figure S3.2

The system has a fixed point at the origin, but it is not simple; this explains why the phase curves in the neighbourhood of the origin behave differently from any of the standard examples.

Solution 3.4

Consider the magnitude and direction of the phase velocity at any point in the phase plane. Changing the sign of *both* components of the phase velocity merely rotates the velocity through 180°, keeping the magnitude the same, as shown in the following figure.

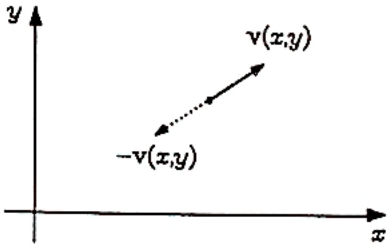


Figure S3.3

Thus the phase curves of the two systems are exactly the same, only the direction of motion along them changes.

Another way of seeing this is to change the independent variable t to $\tau = -t$, so that the second set of equations becomes

$$\frac{dx}{d\tau} = f(x, y), \qquad \frac{dx}{d\tau} = g(x, y),$$

which are now exactly the same as the original equations.

(i) In this example the phase curves are given by

$$x(t) = x_0 e^t, \qquad y(t) = y_0 e^t.$$

The solution to the equations

$$\dot{x} = -x, \qquad \dot{y} = -y$$

is obtained from these merely by substituting $-t$ for t; the phase curves remain unchanged, but now both x and y decrease with increasing t.

(ii) This example is clearly related to the system treated in Exercise 3.3(iii); however, it cannot be obtained from that one just by changing the sign of both components of the phase velocity. But if one follows a change of the sign of the phase velocity with the substitution of $-y$ for y, one does obtain the system of Exercise 3.3(iii). This means that the phase curves for the system in this exercise are the same as those in Figure S3.2, but reflected in the y-axis and with the arrows reversed.

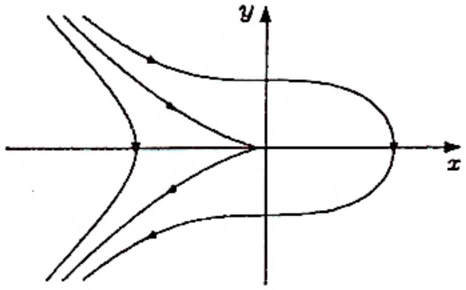


Figure S3.4

Solution 3.5

The form of the equations suggests taking the quotient

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{v_y(x,y)}{v_x(x,y)}.$$

The phase curves are therefore given by the equation usually associated with an autonomous system.

This is the equation derived in Subsection 3.3.2, Equation 3.10. There we said that this equation 'normally' has meaning only if the system is autonomous; here we discuss some special 'abnormal' cases.

In fact, the system can be transformed into an autonomous one simply by defining a new independent variable τ as follows:

$$\frac{d\tau}{dt} = f(t) \qquad \text{or} \qquad \tau = \int dt\, f(t);$$

for then

$$\frac{dx}{d\tau} = \frac{dx}{dt}\frac{dt}{d\tau} = \frac{1}{f(t)}\frac{dx}{dt} = v_x(x,y)$$

and similarly

$$\frac{dy}{d\tau} = v_y(x,y).$$

If f were a function of the time t as well as x and y then the phase curves would still be given by the equation

$$\frac{dy}{dx} = \frac{v_y(x,y)}{v_x(x,y)};$$

but in order to transform to an autonomous system, the new independent variable would have to be obtained by solving the differential equation

$$\frac{dt}{d\tau} = \frac{1}{f(x,y,t)},$$

so that the system would have to be treated as a third-order one.

Solution 3.6

(i) A fixed point is a point $\mathbf{x}_f$ where the phase velocity $\mathbf{v}(\mathbf{x}_f)$ is zero. Alternatively, a fixed point is a state with the property that, if the system is initially in the state, it stays there. Section 3.4

(ii) A fixed point $\mathbf{x}_f$ is said to be an attractor of some motion $\mathbf{x}(t)$ if

$$\lim_{t\to+\infty} \mathbf{x}(t) = \mathbf{x}_f.$$

The fixed point is a repellor of the motion if

$$\lim_{t\to-\infty} \mathbf{x}(t) = \mathbf{x}_f.$$

(iii) A fixed point is strongly (or asymptotically) stable if it is an attractor for all phase curves passing through some neighbourhood of it.

(iv) A fixed point is isolated if there is some neighbourhood of it which contains no other fixed points. Section 3.5

(v) A fixed point of a non-linear system is simple if the corresponding linearised system (the one obtained by linearising about the given fixed point) is simple, that is, if the Jacobian matrix of the non-linear system at the fixed point is non-singular. Subsection 3.6.2

(vi) A simple system is one for which all fixed points are simple; a simple fixed point is defined in the solution to part (v).

Solution 3.7

Separated systems and their fixed points are discussed in Section 3.4.

The system

$$\dot{x} = -x^3, \qquad \dot{y} = -y^3$$

is separated; the origin is a fixed point for each sub-system, and is stable for each. The origin is clearly a fixed point; however, it is not a simple fixed point for either sub-system, so the results of Exercise 3.22 do not apply, and it is not immediately apparent whether or not the origin is strongly stable. We must examine the motion in more detail.

The equations of motion have the solution

$$x = \frac{x_0}{\sqrt{1+2x_0^2 t}}, \qquad y = \frac{y_0}{\sqrt{1+2y_0^2 t}},$$

from which it is clear that, as $t \to \infty$, $\mathbf{x} \to 0$ regardless of the value of (x_0, y_0). The origin is an attractor for all motions, and is therefore certainly strongly stable.

The origin is again clearly a fixed point of the system

$$\dot{x} = -y^2, \qquad \dot{y} = -x^2.$$

To show that it is not strongly stable we have to show that there are points arbitrarily close to it which do not approach it as $t \to \infty$. Now for this system both x and y are always decreasing; and therefore if initially $x_0 < -\varepsilon$, say, where ε is any positive number, then $x(t) < -\varepsilon$ for all $t > 0$, and such a phase point cannot tend to the origin. We may take ε to be as small as we please, provided it is non-zero, from which it follows that there are points arbitrarily close to the origin which escape from it in the motion. The argument could have been based equally well on the assumption $y_0 < -\varepsilon$.

In this case the phase curves are described by the equation $x^3 = y^3 + A$, A being a constant, which also shows that the fixed point is unstable.

Solution 3.8

Section 3.4

A region R_1 of phase space is said to contain the motion from some other region R_2 if every motion originating in R_2 at time $t = 0$ remains within R_1 for all $t \geq 0$. Alternatively, R_1 contains the motion from R_2 if

$$\mathbf{x}(0) \in R_2 \quad \Longrightarrow \quad \mathbf{x}(t) \in R_1 \quad \text{for all } t \geq 0.$$

Solution 3.9

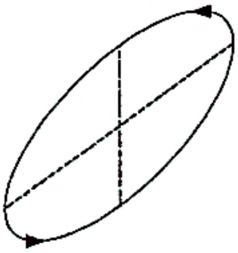
The classification of fixed points of linear systems is described in Section 3.5, with a summary in Subsection 3.5.5.

We need only find the eigenvalues of the matrix $\mathbf{A}$, where $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$; so we need to know only the trace and determinant of $\mathbf{A}$.

(i) $\mathbf{A} = \begin{bmatrix} 1 & -5 \\ 1 & -1 \end{bmatrix}$, $\operatorname{tr} \mathbf{A} = 0$, $\det \mathbf{A} = 4$; so the characteristic equation is

$$\lambda^2 + 4 = 0,$$

giving $\lambda = \pm 2i$. The eigenvalues are purely imaginary, so the fixed point is a centre, and is stable but not strongly so.



Centre
$\lambda = \pm 2i$

Figure S3.5 $\dot{x} = x - 5y$, $\dot{y} = x - y$. The dashed lines are the transforms of the original x- and y-axes in the coordinate system in which $\mathbf{A}$ takes normal form.

(ii) $\mathbf{A} = \begin{bmatrix} -4 & 2 \\ 3 & -2 \end{bmatrix}$, $\operatorname{tr}\mathbf{A} = -6$, $\det\mathbf{A} = 2$; so the characteristic equation is

$$\lambda^2 + 6\lambda + 2 = 0,$$

giving $\lambda = -3 \pm \sqrt{7}$. The eigenvalues are real and both negative, so the fixed point is a stable node.

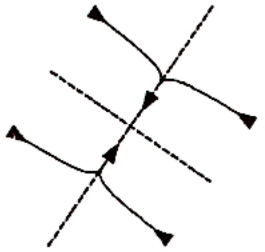


Figure S3.6 $\dot{x} = -4x + 2y$, $\dot{y} = 3x - 2y$. The dashed lines are the transforms of the original x- and y-axes in the coordinate system in which $\mathbf{A}$ takes normal form.

(iii) $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$, $\operatorname{tr}\mathbf{A} = -1$, $\det\mathbf{A} = -3$; so the characteristic equation is

$$\lambda^2 + \lambda - 3 = 0,$$

giving $\lambda = -\frac{1}{2}(1 \pm \sqrt{13})$. The eigenvalues are real and of opposite signs, so the fixed point is a saddle, and is unstable.

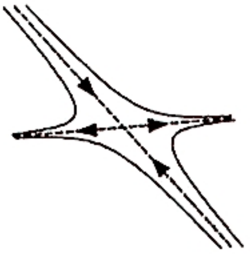


Figure S3.7 $\dot{x} = x + y$, $\dot{y} = x - 2y$. The dashed lines are the transforms of the original x- and y-axes in the coordinate system in which $\mathbf{A}$ takes normal form.

(iv) $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$, $\operatorname{tr}\mathbf{A} = 3$, $\det\mathbf{A} = 3$; so the characteristic equation is

$$\lambda^2 - 3\lambda + 3 = 0,$$

giving $\lambda = \frac{1}{2}(3 \pm i\sqrt{3})$. The eigenvalues are complex, with positive real part, so the fixed point is an unstable spiral.

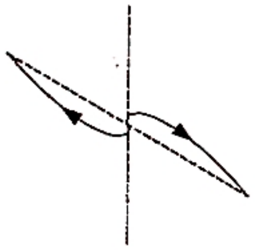


Figure S3.8 $\dot{x} = 2x + y$, $\dot{y} = y - x$. The dashed lines are the transforms of the original x- and y-axes in the coordinate system in which $\mathbf{A}$ takes normal form.

Solution 3.10

(i) The rotation matrix $\mathbf{R}$ is

$$\mathbf{R}(\phi) = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}.$$

In order to prove the result we note that successive rotations through the angles ϕ and θ are equivalent to a single rotation through an angle $\phi+\theta$; thus successive application of the two rotation matrices $\mathbf{R}(\phi)$ and $\mathbf{R}(\theta)$ is equivalent to an application of the matrix $\mathbf{R}(\phi+\theta)$, that is

$$\mathbf{R}(\phi)\mathbf{R}(\theta) = \mathbf{R}(\theta)\mathbf{R}(\phi) = \mathbf{R}(\phi+\theta).$$

Thus

$$\mathbf{R}(\phi)^2 = \mathbf{R}(\phi)\mathbf{R}(\phi) = \mathbf{R}(2\phi)$$

and, by repeated multiplication,

$$\mathbf{R}(\phi)^n = \mathbf{R}(n\phi).$$

Multiplying $\mathbf{S}(\rho,\phi)$ by itself we obtain simply

$$\mathbf{S}(\rho,\phi)^2 = \rho\mathbf{R}(\phi)\rho\mathbf{R}(\phi) = \rho^2\mathbf{R}(2\phi),$$

since a matrix and a scalar commute. Repeated application of this rule then gives

$$\mathbf{S}(\rho,\phi)^n = \mathbf{S}(\rho^n, n\phi) = \rho^n\mathbf{R}(n\phi).$$

(ii) Suppose the result is true for some integer n; then

$$\mathbf{B}^{n+1} = \mathbf{B}\mathbf{B}^n = \begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}\begin{bmatrix}\lambda_1^n & 0\\ 0 & \lambda_2^n\end{bmatrix} = \begin{bmatrix}\lambda_1^{n+1} & 0\\ 0 & \lambda_2^{n+1}\end{bmatrix},$$

the last equality being obtained by direct multiplication of the matrices. Thus if the result is true for any integer n it is also true for $n+1$; as it is clearly true for $n=2$, it is true for all n.

(iii) The calculation of $\mathbf{B}^2$ is straightforward:

$$\mathbf{B}^2 = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{P}^{-1}\mathbf{A}^2\mathbf{P},$$

since $\mathbf{P}^{-1}\mathbf{P} = \mathbf{I}$. Now suppose that the general relation is true for n, then

$$\mathbf{B}^{n+1} = \mathbf{B}\mathbf{B}^n = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}\mathbf{P}^{-1}\mathbf{A}^n\mathbf{P} = \mathbf{P}^{-1}\mathbf{A}^{n+1}\mathbf{P}.$$

Thus, if the relation is true for n then it is also true for $n+1$, and since it is true for $n=2$, it is true for all n.

When $\mathbf{A}$ has two distinct eigenvalues there exists a matrix $\mathbf{P}$ such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ is diagonal, if the eigenvalues are real, or of the form $\rho\mathbf{R}(\phi)$, $\mathbf{R}(\phi)$ being a rotation, if the eigenvalues are complex. Thus for real eigenvalues

$$\mathbf{A}^n = \mathbf{P}\begin{bmatrix}\lambda_1^n & 0\\ 0 & \lambda_2^n\end{bmatrix}\mathbf{P}^{-1}$$

and for complex eigenvalues

$$\mathbf{A}^n = \rho^n\mathbf{P}\mathbf{R}(n\phi)\mathbf{P}^{-1}.$$

Solution 3.11

The eigenvalues λ_1 and λ_2 of the matrix $\mathbf{A}$ satisfy the following relations:

$$\lambda_1+\lambda_2 = \operatorname{tr}\mathbf{A} = a+d > 0 \qquad \text{and} \qquad \lambda_1\lambda_2 = \det\mathbf{A} = ad+b^2 > 0.$$

Note first that $\mathbf{A}$ is necessarily non-singular, so the system is simple and has just one fixed point, at the origin.

If $(\operatorname{tr}\mathbf{A})^2 > 4\det\mathbf{A}$, that is, if $(a-d)^2 > 4b^2$, then the eigenvalues are real and different; since their sum and their product are both positive they must themselves be positive, and the fixed point is an unstable node.

If $(a-d)^2 = 4b^2$ and $b \neq 0$ then the eigenvalues are real, coincident and positive, but the matrix is not a multiple of the identity, and so the fixed point is an unstable improper node. If $b=0$, on the other hand, then $a=d>0$ and the fixed point is an unstable star.

If $(a-d)^2 < 4b^2$ then the eigenvalues are complex, $\lambda = \mu \pm i\nu$ with $\mu = \frac{1}{2}\operatorname{tr}\mathbf{A} > 0$, giving an unstable spiral.

Solution 3.12

The system

$$\dot{\mathbf{x}} = \mathbf{v}(\mathbf{x})$$

is separable if a coordinate system, (X, Y), can be found such that in this representation the equations of motion can be written in the form

Separability is defined at the end of Section 3.4.

$$\frac{dX}{dt} = f(X), \qquad \frac{dY}{dt} = g(Y),$$

where $f(X)$ and $g(Y)$ are functions only of X and Y respectively.

In the case given we have

$$\begin{aligned} \dot{x} &= -\tfrac{3}{5}(X+2Y) + \tfrac{4}{5}(2X - Y) = X - 2Y = \dot{X} + 2\dot{Y}, \\ \dot{y} &= \tfrac{4}{5}(X+2Y) + \tfrac{3}{5}(2X - Y) = 2X + Y = 2\dot{X} - \dot{Y}. \end{aligned}$$

Solving these for $\dot{X}$ and $\dot{Y}$ gives

$$\dot{X} = X, \qquad \dot{Y} = -Y,$$

so the system is separable.

In this coordinate system we may consider the X and Y motions separately. On the X-axis the origin is clearly an unstable fixed point, and on the Y-axis the origin is a stable fixed point. The combined motion is therefore a saddle, with phase diagram as shown in Figure 3.10 (page 59).

Solution 3.13

We need to find the ratio $R(t) = X(t)/(X(t) + Y(t))$, and the easiest way of doing this is by directly solving the equations of motion. We have

$$X(t) = X_0(t)e^{(\lambda-\mu)t}.$$

The equation for $Y(t)$ may be written in the form

$$e^{\nu t}\frac{d}{dt}\left(e^{-\nu t}Y(t)\right) = \mu X_0 e^{(\lambda-\mu)t},$$

which gives, for $\lambda - \mu \neq \nu$,

$$e^{-\nu t}Y(t) = \frac{\mu X_0}{\lambda - \mu - \nu}\left(e^{(\lambda-\mu-\nu)t} - 1\right) + Y_0,$$

whence

$$X(t) + Y(t) = e^{\nu t}\left\{Y_0 - \left(\frac{\mu}{\lambda-\mu-\nu}\right)X_0 + \left(\frac{\lambda-\nu}{\lambda-\nu-\mu}\right)X_0 e^{(\lambda-\mu-\nu)t}\right\}.$$

Thus

$$R(t) = \frac{(\lambda-\mu-\nu)X_0 e^{(\lambda-\mu-\nu)t}}{(\lambda-\mu-\nu)Y_0 - \mu X_0 + (\lambda-\nu)X_0 e^{(\lambda-\mu-\nu)t}}.$$

It follows that if $\lambda > \mu + \nu$ then $R(t) \to (\lambda - \mu - \nu)/(\lambda - \nu)$ as $t \to \infty$, while if $\lambda < \mu + \nu$ then $R(t) \to 0$; in either case the ratio tends to a value which is independent of X_0 and Y_0.

If $\lambda = \mu + \nu$ then the solution becomes

$$\begin{aligned} X(t) &= X_0(t)e^{\nu t}, \\ e^{-\nu t}Y(t) &= Y_0 + t\mu X_0, \\ X(t) + Y(t) &= e^{\nu t}\left\{Y_0 + (1 + t\mu)X_0\right\}e^{\nu t}, \\ R(t) &= \frac{X_0}{Y_0 + (1+t\mu)X_0 e^{\nu t}}, \end{aligned}$$

so that $R(t) \to 0$ as $t \to \infty$.

Of course, if $X_0 = 0$ then $X(t) = 0$ for all t; but then the question is without interest.

The condition for the limiting value of the ratio to be non-zero is thus $\lambda > \mu + \nu$, and the limiting value is $(\lambda - \mu - \nu)/(\lambda - \nu)$.

This system is a linear one; and so is the system with variables X and $X + Y$: in fact, if we set $Z = X + Y$ we have

$$\dot{X} = (\lambda - \mu)X, \qquad \dot{Z} = (\lambda - \nu)X + \nu Z.$$

The ratio $X(t)/Z(t)$ may be regarded as determining the direction of the position vector of the point $(X(t), Z(t))$ in the phase space for this system; in the long term the direction of the phase point tends to that of the eigenvector corresponding to the eigenvalue of largest modulus of the coefficient matrix. The eigenvalues of the coefficient matrix are just its diagonal entries, since it is lower triangular; the corresponding eigenvectors are (scalar multiples of)

$$[0 \quad 1]^T \qquad \text{and} \qquad [\lambda - \mu - \nu \quad \lambda - \nu]^T.$$

(Note that the conditions $\lambda \neq \mu$ and $\nu \neq 0$ ensure that the system is simple.)

Solution 3.14

(i) An isolated fixed point of a non-linear system is not necessarily simple. The system with velocity field $\mathbf{v}(\mathbf{x}) = (x^2, y^2)$ shows this: it has an isolated fixed point at the origin; but the Jacobian matrix of $\mathbf{v}(\mathbf{x})$ at the origin is the zero matrix, which is certainly singular, so the fixed point is not simple.

(ii) A simple fixed point of a non-linear system is necessarily isolated.

The linearisation of systems around their fixed points is discussed in Section 3.6.2; notions of simplicity and isolation are introduced at the beginning of the discussion.

Solution 3.15

The linear classification gives a reliable qualitative picture of the local behaviour of a non-linear system near a fixed point in all cases except when the fixed point is a centre of the linearised system.

This result is stated as the Linearisation Theorem on page 71 of *Unit 3*.

Solution 3.16

The technique required for this exercise is explained in Example 3.4.

(i) For the system $\dot{x} = 1 - xy$, $\dot{y} = y(1 - x)$, the fixed points are the solutions of

$$1 - xy = 0, \quad y(1 - x) = 0 \quad \Longrightarrow \quad x = 1, \quad y = 1.$$

There is thus a single fixed point, at $(1, 1)$. The Jacobian matrix is

$$\begin{bmatrix} -y & -x \\ -y & 1 - x \end{bmatrix},$$

so the matrix $\mathbf{A}$ for the linearised system is

$$\mathbf{A} = \begin{bmatrix} -1 & -1 \\ -1 & 0 \end{bmatrix}.$$

We have $\operatorname{tr}\mathbf{A} = -1$, $\det\mathbf{A} = -1$, so the eigenvalues of $\mathbf{A}$ are $\frac{1}{2}(-1 \pm \sqrt{5})$, which are real and of opposite signs, so the fixed point is a saddle.

(ii) The fixed points of the system $\dot{x} = ye^y$, $\dot{y} = 1 - x^2$, are at the roots of the equations

$$ye^y = 0, \quad 1 - x^2 = 0 \quad \Longrightarrow \quad x = \pm 1, \quad y = 0.$$

There are two fixed points, $(1, 0)$ and $(-1, 0)$. The Jacobian matrix is

$$\begin{bmatrix} 0 & (1 + y)e^y \\ -2x & 0 \end{bmatrix}.$$

At $(1, 0)$ we have

$$\mathbf{A}(1, 0) = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix},$$

with eigenvalues $\pm\sqrt{2}i$, so this fixed point is a centre of the linearised system. It is not possible to say how the non-linear system behaves near this fixed point without further examination; in this case such an analysis is not very difficult and is given in the solution to the next exercise.

For the fixed point at $(-1,0)$ we have

$$\mathbf{A}(-1,0)=\begin{bmatrix}0 & 1\\ 2 & 0\end{bmatrix},$$

with eigenvalues $\pm\sqrt{2}$, so this is a saddle.

(iii) For the system $\dot{x}=2xy-6y-8$, $\dot{y}=y^2-x^2$, the fixed points satisfy

$$y^2-x^2=0 \quad\Longrightarrow\quad x=\pm y$$

and

$$2xy-6y-8=0, \qquad \text{or} \qquad \pm\, 2y^2-6y-8=0.$$

The quadratic $2y^2-6y-8=0$ has roots -1 and 4; the other choice of sign gives a quadratic with no real roots. The fixed points are therefore at $(-1,-1)$ and $(4,4)$. The Jacobian matrix is

$$\begin{bmatrix}2y & 2x-6\\ -2x & 2y\end{bmatrix}.$$

At $(-1,-1)$,

$$\mathbf{A}(-1,-1)=\begin{bmatrix}-2 & -8\\ 2 & -2\end{bmatrix}.$$

The eigenvalues are $-2\pm 8i$, giving a stable spiral.

At $(4,4)$,

$$\mathbf{A}(4,4)=\begin{bmatrix}8 & 2\\ -8 & 8\end{bmatrix}.$$

The eigenvalues are $8\pm 4i$, giving an unstable spiral.

(iv) For the system $\dot{x}=4-4x^2-y^2$, $\dot{y}=3xy$, the fixed points are $(0,\pm2)$ and $(\pm1,0)$. The Jacobian matrix is

$$\begin{bmatrix}-8x & -2y\\ 3y & 3x\end{bmatrix},$$

with the following classification of the fixed points.

At $(0,-2)$ and $(0,2)$ the eigenvalues are $\pm i\sqrt{24}$, giving a centre in each case.

At $(-1,0)$ the eigenvalues are -8 and 3; a saddle.

At $(1,0)$ the eigenvalues are -3 and 8; another saddle.

(v) For the system $\dot{x}=\sin(x+y)$, $\dot{y}=y$, whose fixed points are at $(0,n\pi)$ (where n is an integer), the Jacobian is

$$\begin{bmatrix}\cos(x+y) & \cos(x+y)\\ 0 & 1\end{bmatrix}.$$

For n even, the eigenvalues are coincident, both with value 1, giving an unstable improper node.

For n odd, the eigenvalues are ±1, giving a saddle.

(vi) For the system $\dot{x}=x-y+xy$, $\dot{y}=x-y-y^2$, there are two fixed points, at $(0,0)$ and $(2,-2)$. The Jacobian is

$$\begin{bmatrix}1+y & -1+x\\ 1 & -1-2y\end{bmatrix},$$

and at $(2,-2)$ the eigenvalues are $1\pm\sqrt{5}$, so this is a saddle. But at $(0,0)$, $\det\mathbf{A}=0$, so this fixed point is not simple. The reason for this can be seen in the original equations, the right-hand sides of which are identical if the quadratic terms are ignored. On defining $z=x-y$ and using (z,y) as two new variables, the equations of motion become

$$\dot{z}=zy+2y^2, \qquad \dot{y}=z-y^2,$$

and it is clear from this form that the origin is not a standard type of fixed point.

Solution 3.17

For the system

$$\dot{x} = ye^y, \qquad \dot{y} = 1 - x^2,$$

the nature of the fixed point at $(1,0)$ is best examined by defining $z = x - 1$, so that

$$\dot{z} = ye^y, \qquad \dot{y} = -z(z+2)$$

or
$$\frac{dz}{dy} = -\frac{ye^y}{z(z+2)}.$$

The last equation is separable and can be integrated to give

$$(y-1)e^y + z^2(1 + z/3) = A,$$

for some constant A. For small z and x this equation gives the circles $y^2 + z^2 = A + 1$, confirming that the fixed point in the linear approximation is a centre. For larger z and x we can deduce that the phase curves are closed by noting that both $(y-1)e^y$ and $z^2(1+z/3)$ have a local minimum at the origin, and by using the same argument as given in the solution to Exercise 3.25 (page 184), we see that all phase curves in the neighbourhood of the origin are closed.

Solution 3.18

The fixed points are at the roots of the equations

$$x(1-x^2) = 0 \qquad \text{and} \qquad y = 0,$$

that is $y = 0$, and $x = 0$ and $x = \pm 1$. The Jacobian matrix is

$$\begin{bmatrix} 1 - 3x^2 & 0 \\ 0 & 1 \end{bmatrix}.$$

At $(-1,0)$ and $(1,0)$ we have

$$\mathbf{A}(\pm 1, 0) = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix},$$

with eigenvalues 1 and -2, so these fixed points are saddles. On the other hand, at $(0,0)$ the eigenvalues are real and identical, both being equal to 1, which gives an unstable star.

In order to sketch the phase diagram, as well as using the information from this analysis of the fixed points, we note the following facts. The phase diagram is symmetric with respect to reflections in the x- and y-axes, so it is necessary only to consider the first quadrant. On the lines $x = 0$ and $x = 1$ we have $\dot{x} = 0$, so these are phase curves; on them, $\dot{y} > 0$. The y-axis is also a phase curve. For $0 < x < 1$, $\dot{x} > 0$, while for $x > 1$, $\dot{x} < 0$. This information, taken all together, suggests Figure S3.9.

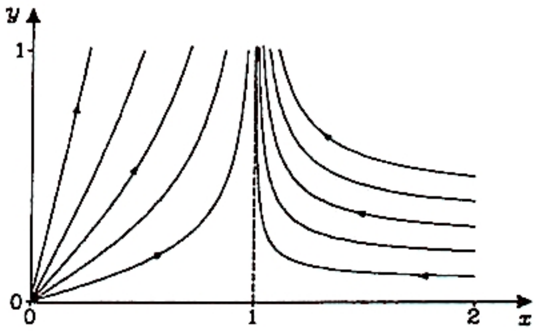


Figure S3.9

Solution 3.19

In both cases we define $y = \dot{x}$.

The technique used in this exercise is described in Example 3.5.

(i) The second-order differential equation becomes

$$\dot{x} = y, \qquad \dot{y} = -(x + x^3 + y),$$

with only one fixed point, at $(0,0)$. The Jacobian matrix is

$$\begin{bmatrix} 0 & 1 \\ -(1+3x^2) & -1 \end{bmatrix},$$

and at $(0,0)$ we have

$$\mathsf{A} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}.$$

This has eigenvalues $\frac{1}{2}(-1 \pm i\sqrt{3})$, so the fixed point is a stable spiral.

(ii) The second-order differential equation becomes

$$\dot{x} = y, \qquad \dot{y} = x^3 - x - y,$$

with fixed points at $(0,0)$, $(-1,0)$ and $(1,0)$. The Jacobian matrix is

$$\begin{bmatrix} 0 & 1 \\ 3x^2 - 1 & -1 \end{bmatrix}$$

and the fixed points are classified as follows.

At $(0,0)$ the eigenvalues are $\frac{1}{2}(-1 \pm i\sqrt{3})$, giving a stable spiral.

At $(-1,0)$ and $(1,0)$ the eigenvalues are 1 and -2, so these are both saddles.

Solution 3.20

Exercise 3.38 is similar.

The fixed point at the origin of the linearised system is a centre. If we convert to polar coordinates we obtain

$$\dot{r} = r^5, \qquad \dot{\theta} = 1.$$

This shows that $\dot{r} > 0$ and $\theta = \theta_0 + t$, so that the phase curves spiral out from the origin, which is therefore strongly unstable.

Solution 3.21

(i) The simultaneous equations

$$x + y + 2 = 0 \qquad \text{and} \qquad x + y + 1 = 0$$

have no roots, so there are no fixed points.

The phase portrait is most easily drawn using the variables x and $z = x + y$, for which the equations become

$$\dot{z} = 2z + 3, \qquad \dot{x} = z + 2.$$

The equation for z is uncoupled from that for x, which is why this representation is convenient. We see that $z = -\frac{3}{2}$ is a fixed point of the z motion, and that on this line x increases uniformly with t, $x = x_0 + \frac{1}{2}t$. For $z > -\frac{3}{2}$, $\dot{z} > 0$, so both z and x are increasing, and for large z we have $dz/dx \simeq 2$; for large negative times these phase curves asymptotically approach the line $z = -\frac{3}{2}$ as $x \to -\infty$. For $z < -\frac{3}{2}$, z decreases; x increases for $-2 < z < -\frac{3}{2}$ and decreases for $z < -2$, and is stationary at $z = -2$. For z large in magnitude and negative we also have $dz/dx \simeq 2$. From this information we can sketch the phase curves in the (x, z)-representation; these are shown in Figure S3.10.

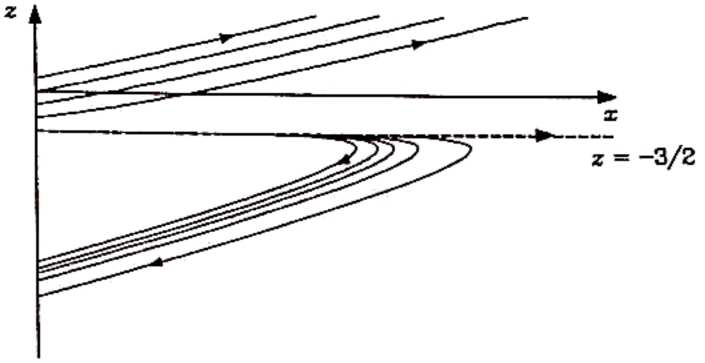


Figure S3.10

The phase curves in the (x, y)-representation can be obtained from this by a simple linear transformation and are shown in Figure S3.11.

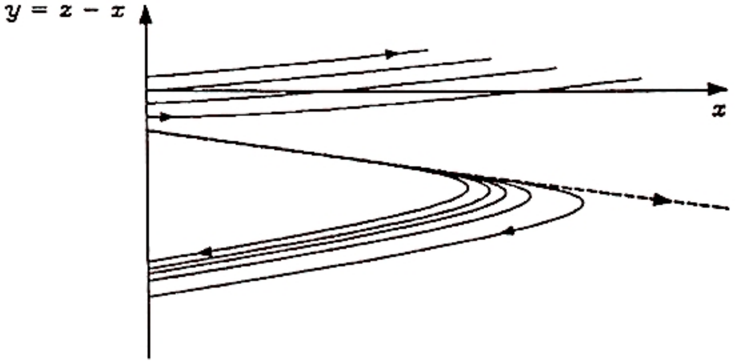


Figure S3.11

(ii) The fixed points are at the simultaneous real roots of

$$x + y = 0 \qquad \text{and} \qquad e^{x+y} = 0,$$

which do not exist, so there are no fixed points.

The phase portrait of this pair of equations can be sketched using the same variable $z = x + y$ as use in part (i). But now it is more convenient to use the (z, y)-coordinates so the equations become

$$\dot{z} = z + e^{z}, \qquad \dot{y} = z.$$

Again the z motion is uncoupled from the y motion. There is a fixed point, z_f, at the root of $z = -e^{z}$; by sketching the graphs of $w = -z$ and $w = e^{z}$, as in Figure S3.12, it is seen that there is only one root and that it is negative.

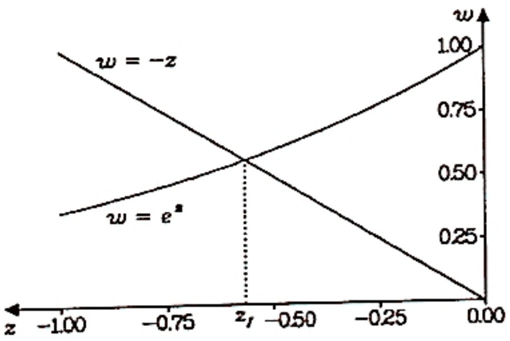


Figure S3.12

The actual value does not matter for a sketch of the phase curves, but it is approximately $z_f = -0.567$. On this line $\dot{y} < 0$, so $y = a + z_f t$ for some constant a. For $z < z_f$, both z and y are decreasing, and for large z, $dz/dy \approx 1$, as e^{-z} is negligible, so the phase curves are at 45° to the y-axis, as shown in the sketch below.

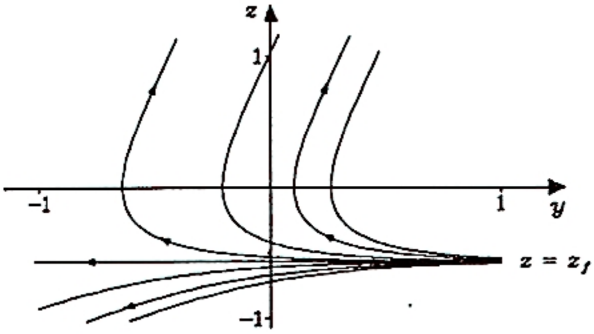


Figure S3.13

For $z > z_f$ the motion is more complicated. First, note that for $z_f < z < 0$, y is decreasing, and that on the y-axis, $\dot{y} = 0$, and for $z > 0$, y is increasing. Secondly, note that for large z

$$\frac{dz}{dt} \approx e^z, \qquad \text{so} \quad z(t) \approx -\ln(A - t),$$

so the z motion terminates at infinity in a finite time. Also we have, on using this approximate solution,

$$\frac{dy}{dt} \approx -\ln(A - t), \qquad \text{so} \quad y \approx B + (A - t)\left(\ln(A - t) - 1\right),$$

and as $z \to \infty$, y approaches the finite value B.

With this information we are able to sketch the phase curves shown in Figure S3.13 above.

Solution 3.22

First convert the equation to the conventional form by defining the new variable $y = \dot{x}$, to give

$$\dot{x} = y, \qquad \dot{y} = F(x, y).$$

At the fixed points, $y = 0$ and x satisfies

$$F(x, 0) = 0.$$

At a fixed point $(a, 0)$ the linearised system has the matrix

$$\mathbf{A}(a, 0) = \begin{bmatrix} 0 & 1 \\ \dfrac{\partial F}{\partial x}(a, 0) & \dfrac{\partial F}{\partial y}(a, 0) \end{bmatrix}.$$

Since $F(x, y)$ is an even function of y, we have $\dfrac{\partial F}{\partial y} = 0$ when $y = 0$, so $\operatorname{tr} \mathbf{A} = 0$ and the eigenvalues of $\mathbf{A}$ are given by

$$\lambda^2 = \frac{\partial F}{\partial x}(a, 0).$$

These are either real and of opposite sign (if $\partial F/\partial x > 0$), producing a saddle; or they are purely imaginary (if $\partial F/\partial x < 0$), giving rise to a centre.

Solution 3.23

For this system both $\dot{x}$ and $\dot{y}$ are zero on the curve $x^2 = y^3$, so that every point on this curve is a fixed point, and arbitrarily close to any fixed point there lies another. That is, the fixed points are not isolated.

Subsection 3.6.2

A fixed point is simple if the Jacobian matrix at the fixed point is non-singular, so we need to evaluate the Jacobian on this line of fixed points. In order to help understand the algebra it is easier to consider the more general set of equations

$$\dot{x} = f(x,y), \qquad \dot{y} = g(x,y)f(x,y);$$

then $f(x,y) = 0$ defines a curve in the phase plane consisting of fixed points. The Jacobian (determinant) of this system at any point is

$$\begin{aligned} J(x,y) &= \begin{vmatrix} f_x & f_y \\ fg_x + gf_x & fg_y + gf_y \end{vmatrix} \\ &= f_x\left(fg_y + f_y g\right) - f_y\left(fg_x + f_x g\right) \\ &= f\frac{\partial(f,g)}{\partial(x,y)}, \end{aligned}$$

where for convenience we use the notation $f_x = \partial f/\partial x$ etc. Note that this is just an example of the product rule

$$\frac{\partial(X,YZ)}{\partial(x,y)} = Y\frac{\partial(X,Z)}{\partial(x,y)} + Z\frac{\partial(X,Y)}{\partial(x,y)}.$$

On the curve of fixed points $f(x,y) = 0$, so $J(x,y) = 0$ and the fixed points are not simple.

In general, a curve of fixed points exists if the components of the velocity function have a common factor, so the equations of motion have the form

$$\dot{x} = f(x,y)F(x,y), \qquad \dot{y} = f(x,y)G(x,y),$$

where $f(x,y)$ is the common factor, and the curve of fixed points is defined by the equation $f(x,y) = 0$. The Jacobian for this system is, by repeated application of the above rule,

$$\frac{\partial(fF,fG)}{\partial(x,y)} = f^2\frac{\partial(F,G)}{\partial(x,y)} + fG\frac{\partial(F,f)}{\partial(x,y)} + fF\frac{\partial(f,G)}{\partial(x,y)} + FG\frac{\partial(f,f)}{\partial(x,y)}.$$

The last term is zero for all (x,y), and the remaining terms are all zero on the curve of fixed points $f(x,y) = 0$. Thus every fixed point on this curve is not simple.

Solution 3.24

The proportionate death rate of the hares is of the form $a + BF$, where $a > 0$ is the proportionate death rate in the absence of any foxes, and $B > 0$ is the rate of increase of the proportionate death rate with F. The proportionate birth rate of the hares is constant, say $a' > 0$. Thus

$$\frac{1}{H}\frac{dH}{dt} = a' - (a + BF) = A - BF \quad \Longrightarrow \quad \frac{dH}{dt} = AH - BHF.$$

Here $A = a' - a$, which we may assume to be positive, since we should expect the hare population to increase if there were no foxes.

The proportionate increase in the fox population due to births depends upon the amount of food, that is, the number of hares. This will take the form CH, since if there are no hares there will be no fox births, where C is a positive constant relating the number of fox births to the availability of food. The proportionate death rate of the fox population is constant, say $D > 0$. Thus we have

$$\frac{1}{F}\frac{dF}{dt} = CH - D \quad \Longrightarrow \quad \frac{dF}{dt} = CHF - DF.$$

The only fixed point is at $F = A/B$ and $H = D/C$ (the velocity field is zero at other points but these are ruled out of consideration by the conditions $H > 0$, $F > 0$). The coefficient matrix for the linearised system is

$$\mathsf{A} = \begin{bmatrix} 0 & -BD/C \\ CA/B & 0 \end{bmatrix},$$

whose eigenvalues are given by the equation

$$\lambda^2 = -AD < 0$$

and are purely imaginary; thus the fixed point is a centre of the linearised system.

Near this fixed point put $F = A/B + f$ and $H = D/C + h$, and expand to first order to obtain the equations

$$\dot{h} = -\frac{BD}{C}f, \qquad \dot{f} = \frac{CA}{B}h,$$

with solution

$$h(t) = \alpha\cos(\sqrt{AD}\,t + \delta), \qquad f(t) = \alpha\frac{C}{B}\sqrt{\frac{A}{D}}\sin(\sqrt{AD}\,t + \delta),$$

where α and δ are constants. Thus (to this approximation) the numbers of foxes and hares execute small periodic oscillations about the fixed point with period $T = 2\pi/\sqrt{AD}$.

In fact, it can be shown that all the phase curves of this system are closed, so that the fixed point is a centre, and the small oscillation approximation is meaningful.

Solution 3.25

The phase curves of the system are given by the solutions of the equation

$$\frac{dH}{dF} = \frac{H(a - bF)}{F(cH - d)}.$$

This is separable and may be written in the form of the two integrals

$$\int dH\left(c - \frac{d}{H}\right) = \int dF\left(\frac{a}{F} - b\right),$$

which may be integrated directly to give

$$cH - d\ln(H) = -(bF - a\ln(F)) + A, \tag{S3.1}$$

where A is a constant. In order to determine the shape of the phase curves from this equation consider the function

$$y(\alpha, \beta; x) = \alpha x - \beta\ln(x),$$

where α and β are positive constants. For small x this is dominated by the logarithmic term, so is large and positive. For large x, as $x \gg \ln(x)$, the first term dominates and again $y(x)$ is large and positive. The derivative $y'(x) = \alpha - \beta/x$ has one zero at $x = \beta/\alpha$, which must be a minimum. This graph is shown in Figure S3.14.

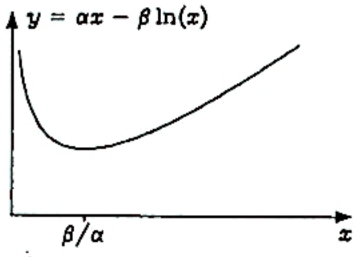


Figure S3.14

From this graph we can construct the shape of the phase curves, for Equation S3.1 shows that H and F satisfy $y(c, d; H) + y(b, a; F) = A$, so that if $y(c, d; H)$ increases then $y(b, a; F)$ must decrease. Figure S3.15 shows how the phase curve is constrained by this equation. Starting at the point (1), at which point H is decreasing and F is increasing, we move to the point (2) and then to (3), at which point $y(H)$ can decrease no further, so F has reached its minimum and must start to increase. We now move back through points (4), (5) to (6) where $y(F)$ can increase no further so H must stop increasing and turn round, to retrace the curve via the points (7), (8) and (9) where once again $y(H)$ can decrease no further so F reaches its maximum. Thus in this way we can trace the path round the phase curve.

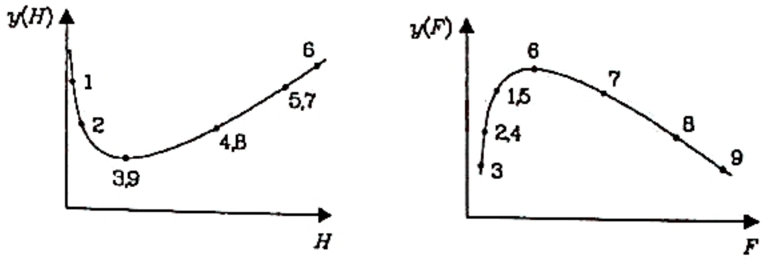


Figure S3.15

This produces the phase curves shown in Figure S3.16, on which are shown the extreme points (3), (6) and (9) of Figure S3.15.

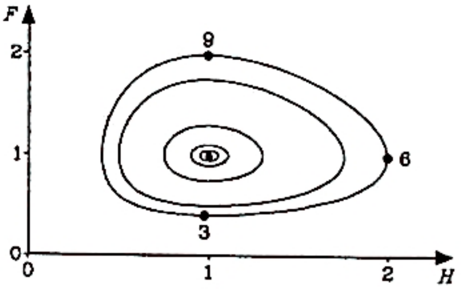


Figure S3.16

Thus all motion is periodic. A typical example of the graph of $F(t)$ is shown in Figure S3.17, and if you look carefully you will see that the periods of the two curves shown are slightly different; this is expected, as non-linear effects usually make the period of the motion amplitude-dependent.

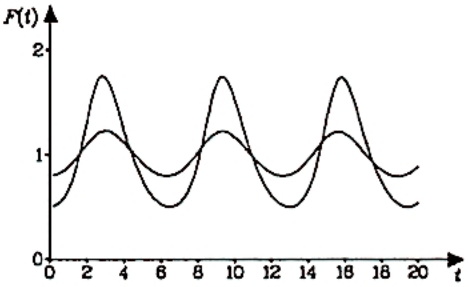


Figure S3.17

Phase space comprises a set of concentric closed curves, which for convenience we depict by concentric circles in Figure S3.18, and a *sudden* change in the population of one of the species, due to circumstances not allowed for in the equations of motion, will result in the motion leaving one phase curve and starting up on another.

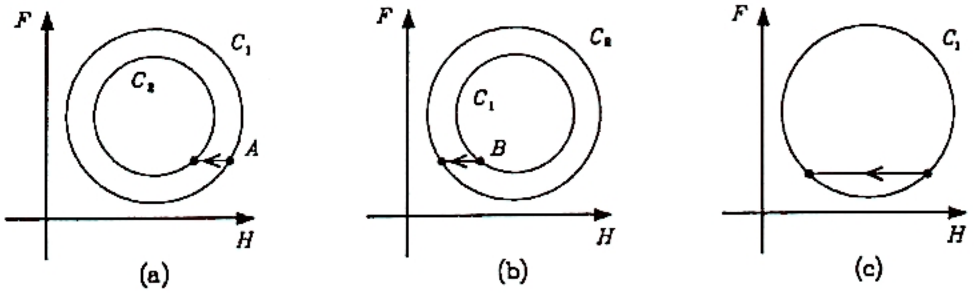


Figure S3.18

Thus if the initial phase curve is C_1 of Figure S3.18(a) and the decrease in the hare population occurs at the point A, then the motion will start again on the curve C_2, that is the population of both species will fluctuate less and the populations will be nearer the values at the fixed point. It can be seen that this effect can be achieved only if the change occurs above the H-turning points, but precisely how far above depends upon a variety of factors, including the magnitude of the change and the curvature of the phase curves. If the change occurs below these turning points, for instance at B in Figure S3.18(b), then the motion will move to an outer curve and the populations will oscillate with larger amplitude.

Finally, we note that if the change occurs at precisely the right place, as shown in the third figure, it merely results in setting the motion back onto the original phase curve.

Solution 3.26

Define a new variable $y = \dot{\psi}$ to give the system of equations

$$\dot{\psi} = y, \qquad \dot{y} = -\alpha y - \omega^2 \sin\psi,$$

which has fixed points at $y = 0$ and $\sin\psi = 0$, that is, $\psi = n\pi$, $n = 0, \pm 1, \pm 2, \ldots$. The even values of n correspond to the pendulum pointing vertically downwards, so we should expect these fixed points to be stable, while the odd values of n correspond to the pendulum pointing vertically upwards, so we should expect these fixed points to be unstable.

The coefficient matrix for the linearised system is

$$\begin{bmatrix} 0 & 1 \\ -\omega^2\cos\psi & -\alpha \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \mp\omega^2 & -\alpha \end{bmatrix},$$

where the upper sign is chosen for a fixed point with n even, the lower sign for one with n odd. The eigenvalue equation is then

$$\lambda^2 + \alpha\lambda \pm \omega^2 = 0.$$

For the fixed points corresponding to the pendulum pointing downwards, with n even, the eigenvalues are

$$\tfrac{1}{2}\left(-\alpha \pm \sqrt{\alpha^2 - 4\omega^2}\right),$$

so the nature of the fixed point depends upon whether or not $\alpha > 2\omega$. The three possibilities are as follows.

- If $\alpha > 2\omega$ then both eigenvalues are real and negative, so the fixed point is a stable node: this is known as strong damping.
- If $\alpha < 2\omega$ then the eigenvalues are complex, with negative real part, so the fixed point is a stable spiral: this is the case of weak damping, when the solution (for ψ) is oscillatory with decreasing amplitude.
- If $\alpha = 2\omega$, the eigenvalues are real, equal and negative giving a stable improper node: this is critical damping.

For the fixed points corresponding to the pendulum pointing vertically upwards the eigenvalues are

$$\tfrac{1}{2}\left(-\alpha \pm \sqrt{\alpha^2 + 4\omega^2}\right),$$

which are real, distinct and of opposite sign, so these fixed points are saddles.

In the case $\alpha = 0$ the eigenvalues of the stable fixed points become purely imaginary, so these fixed points become centres. The unstable fixed points, however, remain unchanged in type.

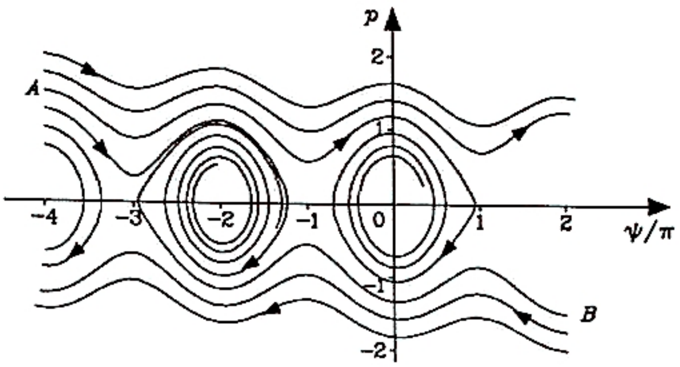


Figure S3.19

Some typical phase curves are shown in Figure S3.19. All phase curves are similar and represent motion circling clockwise if $\dot{\psi} > 0$ in the distant past, or anti-clockwise in the opposite case. Consider curve A in the figure, which starts from the far left and represents a rotational motion which is slowly damped; eventually it loses so much energy from frictional forces that it has insufficient to rotate all the way round, and starts to spiral in towards the origin, or more generally to one of the equivalent points $(2n\pi, 0)$. The only variation of this theme is shown by curve B, which differs from curve A simply because the rotation is anti-clockwise.

Solution 3.27

If a function $f(x, y)$ is constant along all phase curves then its total differential with respect to time is zero. On using the chain rule this gives

$$0 = \frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = \frac{\partial f}{\partial x}u(x, y) + \frac{\partial f}{\partial y}v(x, y).$$

In the example given, the phase curves are solutions of the differential equation

$$\frac{dy}{dx} = \frac{\ln x}{xy}.$$

This is separable, and can be written in the form

$$\int dy\, y = \int dx\, \frac{\ln x}{x}, \qquad \text{giving} \quad y^2 - (\ln x)^2 = A,$$

where A is a constant of integration. Thus the function

$$f(x, y) = y^2 - (\ln x)^2$$

is a constant along all phase curves and, from the result obtained in the first part of the question, it must satisfy the partial differential equation

$$xy\frac{\partial f}{\partial x} + \ln x\frac{\partial f}{\partial y} = 0.$$

Solution 3.28

(i) If $\mathbf{y}(t) = \mathbf{z}(t + T)$ then

$$\frac{d\mathbf{y}}{dt}(t) = \frac{d\mathbf{z}}{dt}(t + T) = \mathbf{A}(t + T)\mathbf{z}(t + T) = \mathbf{A}(t)\mathbf{y}(t).$$

(ii) From the first part of the question, for any integer k, $\mathbf{z}(t + kT)$ is a solution, and therefore

$$\mathbf{z}(t + kT) = \mathbf{X}(t)\mathbf{z}(kT).$$

Thus

$$\mathbf{z}((k + 1)T) = \mathbf{X}(T)\mathbf{z}(kT),$$

so that

$$\mathbf{z}((k + 1)T) = \mathbf{X}(T)\mathbf{X}(T)\mathbf{z}((k - 1)T), \quad \text{etc.}$$

Thus it follows that

$$\mathbf{z}(nT) = \mathbf{X}(T)^n \mathbf{z}(0) = \mathbf{X}(T)^n \mathbf{a}.$$

If the standard form of $\mathbf{X}(T)$ is $\mathbf{B}$, so that $\mathbf{X}(T) = \mathbf{P}^{-1}\mathbf{B}\mathbf{P}$, then

$$\mathbf{P}\mathbf{z}(nT) = \mathbf{B}^n(\mathbf{P}\mathbf{a}).$$

Thus if the eigenvalues λ_1 and λ_2 of $\mathbf{X}(T)$ are real,

$$\mathbf{B}^n = \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix}$$

and the solution grows without bound if either $|\lambda_1| > 1$ or $|\lambda_2| > 1$, and decays to zero if both $|\lambda_1| < 1$ and $|\lambda_2| < 1$.

If the eigenvalues are complex then

$$\mathbf{B}^n = \rho^n \begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix},$$

where $\rho = |\lambda_1| = |\lambda_2|$. So if $\rho > 1$ the solution grows without bound, and if $\rho < 1$ the solution decays, whereas if $\rho = 1$, $\mathbf{B}^n$ remains non-zero and finite and the points $\mathbf{P}\mathbf{z}(nT)$, $n = 0, 1, 2, \ldots$, lie on the unit circle, so the points $\mathbf{z}(nT)$ lie on an ellipse.

Solution 4.1

Hamilton's equations are

$$\dot{q} = \frac{\partial H}{\partial p}, \qquad \dot{p} = -\frac{\partial H}{\partial q}.$$

Hamilton's equations are given in Section 4.2, Equations 4.14. The conditions for a system to be Hamiltonian are derived in Exercise 4.6.

The system

$$\dot{q} = Q(q, p, t), \qquad \dot{p} = P(q, p, t)$$

is Hamiltonian if and only if there exists a function $H(q, p, t)$, the Hamiltonian, satisfying the equations

$$Q(q, p, t) = \frac{\partial H}{\partial p}(q, p, t), \qquad P(q, p, t) = -\frac{\partial H}{\partial q}(q, p, t).$$

The necessary condition for this is obtained by cross-differentiation, and is

$$\frac{\partial Q}{\partial q} = -\frac{\partial P}{\partial p}.$$

This is also a sufficient condition.

Solution 4.2

We write the equations of motion as $\dot{x} = X(x, y, t)$, $\dot{y} = Y(x, y, t)$. For the examples of this exercise, using the test from the previous one, we have the following results. (For convenience we write X_x and Y_y for the partial derivatives of X and Y.)

	X	Y	X_x	$-Y_y$	$H(x, y, t)$
(i)	xy	$-xy + \sin(yt)$	y	$x - t\cos(yt)$	—
(ii)	x^2y	$x^3 - xy^2$	$2xy$	$2xy$	$\frac{1}{4}x^2(2y^2 - x^2)$
(iii)	yt	xt^2	0	0	$\frac{1}{2}(y^2t - x^2t^2)$
(iv)	$-y^2$	$-x^2$	0	0	$\frac{1}{3}(x^3 - y^3)$
(v)	$x - 5y$	$x - y$	1	1	$-\frac{1}{2}(x^2 - 2xy + 5y^2)$
(vi)	$x + y$	$x - 2y$	1	2	—

Solution 4.3

Using the same notation as in the previous solution, so that

$$X(x,y) = y^2, \qquad Y(x,y) = -\frac{x}{2y},$$

we have

$$\frac{\partial X}{\partial x} = 0, \qquad \frac{\partial Y}{\partial y} = \frac{x}{2y^2},$$

and consequently the system is not Hamiltonian in this coordinate system.

Making the substitution, from the first equation of motion we have $\dot{q} = p$. Also,

$$\dot{p} = 2y\dot{y} = -x = -q.$$

These are the same equations of motion as those given by the Hamiltonian

$$H(q,p) = \tfrac{1}{2}(p^2 + q^2).$$

This simple and important Hamiltonian is discussed in Example 4.2. It is the Hamiltonian for the linear oscillator.

Solution 4.4

In the first case we have

$$\dot{q}_1 = \frac{\partial H}{\partial p_1} = \frac{p_1}{m}, \qquad \dot{p}_1 = -\frac{\partial H}{\partial q_1} = -V'(q_1) - F\sin\omega t.$$

The relation between Hamilton's and Newton's equations is described in Section 4.2.

On differentiating the first equation and substituting into the second, we obtain

$$m\ddot{q}_1 + V'(q_1) = -F\sin\omega t.$$

In the second case Hamilton's equations are

$$\dot{q}_2 = \frac{p_2}{m} + \frac{F}{m\omega}\cos\omega t, \qquad \dot{p}_2 = -V'(q_2).$$

Using the same method, we obtain the same equation of motion for q_2.

In the third case we have

$$\dot{q}_3 = \frac{p_3}{m}, \qquad \dot{p}_3 = -V'\left(q_3 + \frac{F}{m\omega^2}\sin\omega t\right),$$

so by differentiating the first equation and substituting in the second we obtain the second-order differential equation

$$m\ddot{q}_3 + V'\left(q_3 + \frac{F}{m\omega^2}\sin\omega t\right) = 0.$$

If we now define $Q = q_3 + (F/m\omega^2)\sin\omega t$, so that

$$\ddot{Q} = \ddot{q}_3 - \frac{F}{m}\sin\omega t,$$

we obtain the same second-order equation as before.

Solution 4.5

(i) A sketch of a suitably shaped wire, together with the forces acting on the bead, is shown in Figure S4.1.

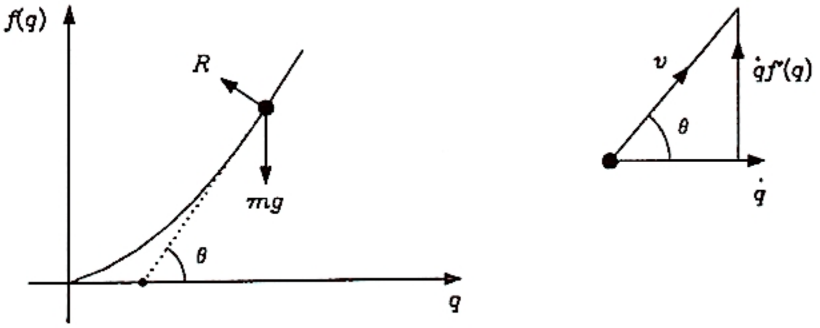


Figure S4.1 Diagram showing the forces acting on a bead sliding smoothly on a wire. The force of reaction is R and is perpendicular to the wire.

From the expression $(q, f(q))$ for the bead's position, we see that its velocity has a component $\dot{q}$ along the q-axis and a vertical component $f'(q)\dot{q}$. This is shown in the inset of the figure. The components of the bead's acceleration are therefore $\ddot{q}$ along the q-axis and

$$\frac{d}{dt}\left(\dot{q}f'(q)\right) = f'(q)\ddot{q} + f''(q)\dot{q}^2$$

vertically. We take the component of the equations of motion in the tangential direction, to avoid introducing the reaction. The force acting along the tangent is $F = -mg\sin\theta$, where θ is the angle between the tangent and the q-axis, as shown in Figure S4.1, so that

$$\tan\theta = f'(q).$$

The tangential component of the acceleration is

$$\ddot{q}\cos\theta + \frac{d}{dt}\left(\dot{q}f'(q)\right)\sin\theta$$

so that the tangential component of Newton's equations of motion can be written in the form

$$m\left(\ddot{q}\cos\theta + \left(f'(q)\ddot{q} + f''(q)\dot{q}^2\right)\sin\theta\right) = -mg\sin\theta,$$

which is simplified by dividing by $\cos\theta$ and using the above relation between $\tan\theta$ and $f'(q)$ to give

$$\left(1 + f'(q)^2\right)\ddot{q} + f'(q)f''(q)\dot{q}^2 + gf'(q) = 0.$$

(ii) For the given Hamiltonian the equations of motion are

$$\dot{q} = \frac{p}{m\sigma(q)^2}, \qquad \dot{p} = -mgf'(q) + \frac{p^2\sigma'(q)}{m\sigma(q)^3},$$

which may be combined to give

$$\frac{d}{dt}\left(\dot{q}\sigma(q)^2\right) = -gf'(q) + \sigma(q)\sigma'(q)\dot{q}^2.$$

Now

$$\sigma(q)^2 = 1 + f'(q)^2 \quad \Longrightarrow \quad \sigma(q)\sigma'(q) = f'(q)f''(q),$$

so the equation just obtained is exactly the same as the equation of motion derived in the first part of the question.

Solution 4.6

On putting $\dot{q} = y$ the equation of motion becomes

$$\dot{q} = y, \qquad \dot{y} = F(q) - y^2G(q).$$

In order for this to be Hamiltonian we require that

$$\frac{\partial}{\partial y}\left(F(q) - y^2G(q)\right) = -\frac{\partial}{\partial q}(y),$$

and this is clearly not true (unless G is identically zero). Thus the system is not Hamiltonian in these coordinates.

Differentiating the equation for p gives

$$\dot{p} = \ddot{q}f(q) + \dot{q}^2f'(q) = f(q)\left(F(q) + G(q)\dot{q}^2\right)$$

since $f' = 2Gf$. Thus if the system is to be Hamiltonian then

$$\dot{p} = f(q)F(q) + \frac{G(q)p^2}{f(q)} = -\frac{\partial H}{\partial q},$$

and also

$$\dot{q} = \frac{p}{f(q)} = \frac{\partial H}{\partial p}.$$

From the second of these,

$$H(q,p) = \frac{p^2}{2f(q)} + V(q)$$

for some function $V(q)$ of q only. Differentiation of this with respect to q gives

$$\frac{\partial H}{\partial q} = V'(q) - \frac{f'(q)p^2}{2f(q)^2} = V'(q) - \frac{G(q)p^2}{f(q)}.$$

On comparing the result with the first equation of motion we find

$$V'(q) = -f(q)F(q),$$

which is the required result.

(The previous exercise is a particular case of this result.)

Solution 4.7

Conservative Hamiltonian systems are defined in Subsection 4.3.1.

A Hamiltonian system is called conservative if the value of the Hamiltonian function is constant along the system's phase curves. Thus for a conservative system the Hamiltonian has the same value throughout a motion as it does initially. The phase curves are therefore the level curves or contours of the Hamiltonian function. An autonomous Hamiltonian system is automatically conservative.

The rate of change of the Hamiltonian is

$$\frac{dK}{dt} = H\frac{df}{dt} + f\frac{dH}{dt} = H\frac{df}{dt} + f\left(\frac{\partial H}{\partial q}\dot{q} + \frac{\partial H}{\partial p}\dot{p}\right),$$

but Hamilton's equations of motion are

$$\dot{q} = \frac{\partial K}{\partial p} = f(t)\frac{\partial H}{\partial p} \qquad \text{and} \qquad \dot{p} = -\frac{\partial K}{\partial q} = -f(t)\frac{\partial H}{\partial q},$$

and substitution into the above equations gives

$$\frac{dK}{dt} = H(q,p)\frac{df}{dt} \neq 0.$$

Thus $K(q,p,t)$ is not constant along most phase curves.

On using the new variable $\tau = \int dt\, f(t)$, and the chain rule, we find that for any function $A(q,p,t)$,

$$\frac{dA}{dt} = \frac{dA}{d\tau}\frac{d\tau}{dt} = f(t)\frac{dA}{d\tau},$$

so we may rewrite Hamilton's equations of motion in the form

$$\frac{dq}{d\tau} = \frac{\partial H}{\partial p}, \qquad \frac{dp}{d\tau} = -\frac{\partial H}{\partial q}.$$

These are autonomous, so the system is conservative if the new time variable is used.

Solution 4.8

If $f(q,p) = F(H(q,p))$ then the chain rule for partial differentiation gives

$$\frac{\partial f}{\partial q} = F'(H)\frac{\partial H}{\partial q} \qquad \text{and} \qquad \frac{\partial f}{\partial p} = F'(H)\frac{\partial H}{\partial p},$$

so that

$$\frac{df}{dt} = \frac{\partial f}{\partial q}\dot{q} + \frac{\partial f}{\partial p}\dot{p} = F'(H)\left(\frac{\partial H}{\partial q}\dot{q} + \frac{\partial H}{\partial p}\dot{p}\right) = 0,$$

the last step following from Hamilton's equations. Thus $f(q,p) = F(H(q,p))$ is constant along all phase curves. The case of the square of the Hamiltonian follows from this general result on taking $F(z) = z^2$.

Solution 4.9

We use the standard test. The equations of motion are of Hamiltonian form if there exists a Hamiltonian function, $H(x,y)$, such that

Linear Hamiltonians are considered in Subsection 4.4.2.

$$\dot{x} = ax + by = \frac{\partial H}{\partial x}, \qquad \dot{y} = cx + dy = -\frac{\partial H}{\partial y};$$

then $H(x,y)$ exists if and only if

$$\frac{\partial^2 H}{\partial y \partial x} = a = \frac{\partial^2 H}{\partial x \partial y} = -d, \qquad \text{so that} \quad a + d = \operatorname{tr} \mathbf{A} = 0.$$

In this case the eigenvalues of $\mathbf{A}$ satisfy the equation

$$\lambda^2 + \det \mathbf{A} = 0,$$

and the fixed point at the origin is a centre if $\det \mathbf{A} > 0$ and a saddle if $\det \mathbf{A} < 0$.

Solution 4.10

(i) A separatrix is a set of phase curves associated with a fixed point which is a saddle. It consists of those phase curves which are attracted or repelled by the saddle, and comprises some of the phase curves with the same energy as the saddle itself. A separatrix has the property of dividing the phase space into regions in which the motion has markedly different characteristics: it may form the boundary between one region in which all motion is bounded and another in which the phase curves escape to infinity, for example.

Separatrixes are first introduced in Example 4.3, the linear repulsive force. A more complicated case is described in Subsection 4.5.2, a cubic potential.

(ii) The quartic potential and some representative phase curves are drawn in Figures S4.2 and S4.3; these have a similar shape to those of the inverted quadratic potential dealt with in Example 4.3, page 103, but with one very important difference. The force is not linear but is proportional to q^3, so near the origin it is much smaller than the linear force, and at large values of q it is much larger. Near the origin the phase curves are flatter than the hyperbola of the quadratic potential, so the momentum changes little, but for large q they increase more rapidly.

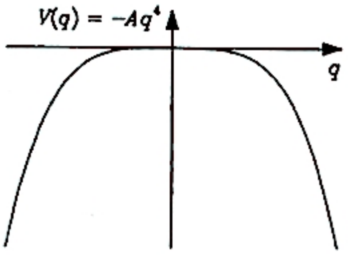


Figure S4.2 Graph of the potential $V(q) = -Aq^4$, $A > 0$.

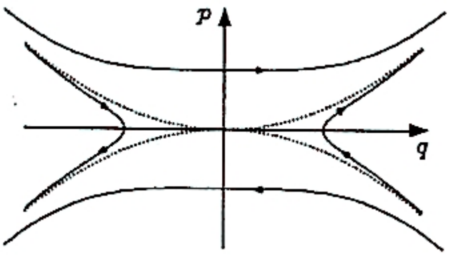


Figure S4.3 Representative phase curves for the Hamiltonian $H(q,p) = \frac{1}{2}p^2 - Aq^4$, $A > 0$. The dotted line is the separatrix of energy $E = 0$.

The energy equation is

$$E = \frac{p^2}{2m} - Aq^4.$$

On the separatrix, $E = 0$, so its equation is

$$p = \pm\left(\sqrt{2Am}\right) q^2,$$

and each of its branches is parabolic.

Hamilton's equation for motion on the branch of the separatrix with q and p positive is

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m} = q^2\sqrt{\frac{2A}{m}} \qquad (E = 0),$$

which is separable, and so may be integrated directly to give

$$\int_{q_0}^{q} \frac{dq}{q^2} = t\sqrt{\frac{2A}{m}} \qquad \text{or} \qquad q = \frac{q_0}{1 - \tau q_0}, \quad \tau = t\sqrt{\frac{2A}{m}}.$$

The motion therefore terminates at $\tau = 1/q_0$.

This is essentially the same as the behaviour of a first-order system with a quadratic velocity function.

For $q < 0$ and $p < 0$ we need only change the sign of t in this equation to obtain the motion

$$q = \frac{q_0}{1 + \tau q_0}.$$

(iii) The energy equation is now

$$E = \frac{p^2}{2m} - Aq^4, \qquad A > 0.$$

The equation of motion can be written as

$$\dot{q} = \sqrt{\varepsilon + q^4}\sqrt{\frac{2A}{m}}, \qquad \text{where} \quad E = A\varepsilon,$$

which, again, is separable. The solution can be written in terms of an integral:

$$\int_{q_0}^{q} \frac{dq}{\sqrt{\varepsilon + q^4}} = t\sqrt{\frac{2A}{m}}.$$

We now need to show that $q(t)$ becomes infinite at some finite time, that is, we need to show that the integral

$$\int_{q_0}^{\infty} \frac{dq}{\sqrt{\varepsilon + q^4}}$$

is finite. For this we note that for $\varepsilon > 0$,

$$\frac{1}{\sqrt{\varepsilon + q^4}} < \frac{1}{q^2},$$

so that

$$\int_{q_0}^{q} \frac{dq}{\sqrt{\varepsilon + q^4}} \le \int_{q_0}^{q} \frac{dq}{q^2},$$

and as the last integral is just the one dealt with in the previous exercise, it follows that the result is true.

Compare end-of-block Exercises 2.4 and 2.13.

Solution 4.11

The potential function, $V(q) = \frac{1}{4}q^4 - \frac{1}{2}q^2$, has stationary points at the roots of $V'(q) = q(q^2 - 1) = 0$, that is at $q = 0$ and $q = \pm 1$. Furthermore, $V''(0) = -1$ and $V''(\pm 1) = 2$, so the potential has a local maximum at $q = 0$ and local minima at $q = \pm 1$. For large $|q|$, $V(q)$ increases monotonically, as q^4. This is all summarised in Figure S4.4, showing the potential graph.

The technique required to solve problems of this kind is exemplified in Subsections 4.5.2 and 4.5.3.

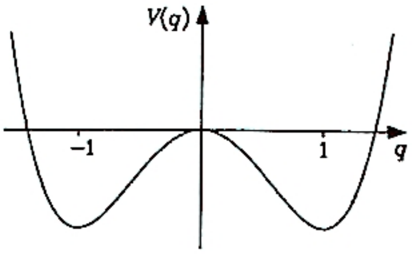


Figure S4.4 Graph of the potential $V(q) = q^4/4 - q^2/2$.

From this figure we see that all motion is bounded and oscillatory, or in other words librational, but that there are three invariant regions of phase space: for energies $E > 0$ there is one region, but for energies $-1/4 \leq E < 0$ there are two regions separated by a potential barrier; for energies $E < -1/4$ no motion is possible. Between these regions and the positive energy region lies the separatrix, of energy $E_s = 0$, with equation

$$\tfrac{1}{2}p^2 + \tfrac{1}{4}q^2(q^2 - 2) = 0 \qquad \text{or} \qquad p_s(q) = \pm q\sqrt{1 - \tfrac{1}{2}q^2}.$$

Representative phase curves in these three regions are shown in Figure S4.5.

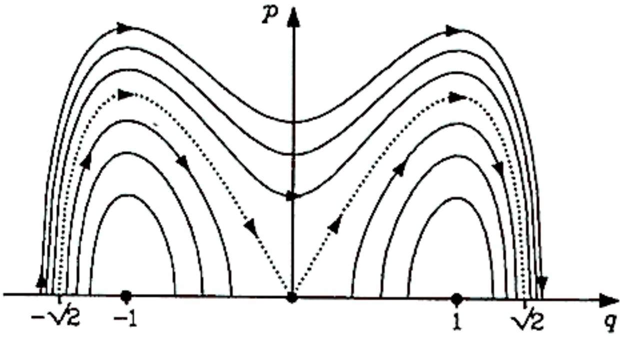


Figure S4.5 Representative phase curves of the Hamiltonian $H(q,p) = p^2/2 + q^4/4 - q^2/2$, with the dots showing the fixed points and the dotted lines the separatrixes.

The two stable fixed points are at $(\pm 1, 0)$ and we obtain an approximation to the motion in their neighbourhoods by making a Taylor expansion of the potential; as the system is symmetric about $q = 0$ it is necessary to deal only with one fixed point. Put $q = 1 + x$ and expand $V(q)$ in powers of x, to obtain

$$\begin{aligned} V(1+x) &= V(1) + xV'(1) + \tfrac{1}{2}x^2V''(1) + O(x^3) \\ &= -\tfrac{1}{4} + x^2 + O(x^3). \end{aligned}$$

Now ignore all but the quadratic terms, to obtain the approximate Hamiltonian

$$H(x,p) = \tfrac{1}{2}p^2 + x^2,$$

which is of linear oscillator type, with the equation of motion

$$\ddot{x} + 2x = 0.$$

The period of small oscillations about the stable fixed points is thus $T = \pi\sqrt{2}$.

Solution 4.12

For the Hamiltonian

$$H(q,p) = \frac{p^2}{2m} + V(q),$$

on the phase curve of energy E we have

$$p(q,E) = \pm\sqrt{2m(E - V(q))}.$$

The calculation of periods is described in Subsection 4.6.5. The case treated there is motion in a circle, but the technique (for librations) is general.

Hamilton's equations give

$$\dot{q} = \frac{p}{m},$$

and so for $\dot{q} > 0$,

$$\dot{q} = \frac{2}{m}\sqrt{E - V(q)}.$$

The time to travel between q_1 and q_2, with $q_2 > q_1$, is therefore

$$T = \int_{q_1}^{q_2} \frac{dq}{\dot{q}} = \sqrt{\frac{m}{2}} \int_{q_1}^{q_2} \frac{dq}{\sqrt{E - V(q)}}.$$

Consider the case of periodic motion, with the motion bounded between $q_1(E)$ and $q_2(E)$, where $q_1(E)$ and $q_2(E)$ are roots of the equation $V(q) = E$, with $q_2(E) > q_1(E)$. For each value of q there are two values of p for points on the phase curve, corresponding to the choice of sign in the square root, and therefore two branches of the phase curve to be taken into account. The time to travel between q_1 and q_2 on the branch with $\dot{q} > 0$ is

$$T_1(E) = \int_{q_1}^{q_2} \frac{dq}{\dot{q}} = \sqrt{\frac{m}{2}} \int_{q_1}^{q_2} \frac{dq}{\sqrt{E - V(q)}},$$

and the time taken to travel between q_2 and q_1 on the branch with $\dot{q} < 0$ is

$$T_2(E) = \int_{q_2}^{q_1} \frac{dq}{\dot{q}} = -\sqrt{\frac{m}{2}} \int_{q_2}^{q_1} \frac{dq}{\sqrt{E - V(q)}} = T_1(E).$$

The period is therefore

$$\begin{aligned} T(E) &= T_1(E) + T_2(E) \\ &= \sqrt{2m} \int_{q_1(E)}^{q_2(E)} \frac{dq}{\sqrt{E - V(q)}}. \end{aligned}$$

Solution 4.13

The force and the potential are related by

$$F(r) = -\frac{dV}{dr}, \qquad \text{so} \quad V(r) = -\int dr \left(-kr + \frac{\alpha}{r^3}\right) = \tfrac{1}{2}\left(kr^2 + \frac{\alpha}{r^2}\right) + A,$$

for some constant A. But the fixed point is the root of $F(r) = 0$, which is $r = (\alpha/k)^{1/4}$, so in order that the potential should vanish there we must have $A = -\sqrt{\alpha k}$, and

$$V(r) = \tfrac{1}{2}\left(r\sqrt{k} - \frac{\sqrt{\alpha}}{r}\right)^2.$$

The graph of this potential is shown in Figure S4.6; notice how it rises more steeply for small r than for large r and how this affects the phase curves shown in Figure S4.7. An analysis of the turning points shows this behaviour. The turning points for motion with energy E are the roots of $V(r) = E$; the turning point with larger r is given by

$$\begin{aligned} r_2(E) &= \frac{\sqrt{2E} + \sqrt{2E + 4\sqrt{\alpha k}}}{2\sqrt{k}} \\ &\simeq \sqrt{\frac{2E}{k}} \qquad \text{for large } E. \end{aligned}$$

The turning point with smaller r is

$$\begin{aligned} r_1(E) = \frac{-\sqrt{2E} + \sqrt{2E + 4\sqrt{\alpha k}}}{2\sqrt{k}} &= \frac{2\sqrt{\alpha}}{\sqrt{2E + 4\sqrt{k\alpha}} + \sqrt{2E}} \\ &\simeq \sqrt{\frac{\alpha}{2E}} \qquad \text{for large } E. \end{aligned}$$

This shows that for large energies the inner turning point, $r_1(E)$, changes much more slowly than does the outer turning point, $r_2(E)$.

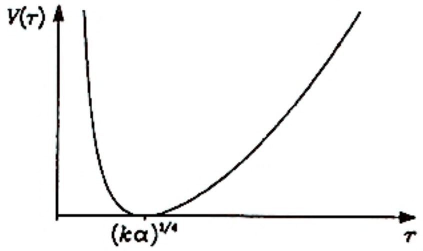


Figure S4.6 Graph of the potential $V(r) = \frac{1}{2}\left(r\sqrt{k} - \sqrt{\alpha}/r\right)^2$.

There is only one fixed point, and it is clear that at this point the potential has a minimum, so it is a stable fixed point. Since the potential tends to infinity as $r \to 0$ and ∞, all motion is bound and periodic. Some representative phase curves are shown in Figure S4.7. Notice that they are closer together at small r, where the potential changes faster with r, so that there the particle is moving faster.

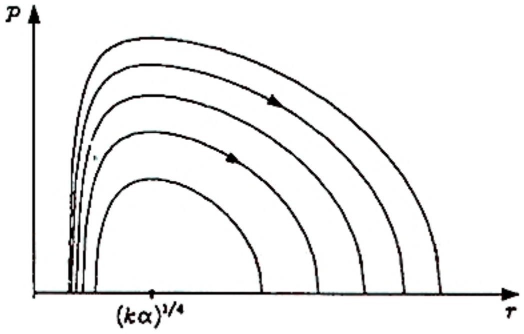


Figure S4.7 Representative phase curves for the Hamiltonian
$H(r,p) = p^2/2 + V(r)$, with $V(r) = \frac{1}{2}\left(r\sqrt{k} - \sqrt{\alpha}/r\right)^2$.

The period can be expressed as an integral as follows:

$$\begin{aligned}
T(E) &= \sqrt{2m}\int_{r_1(E)}^{r_2(E)} \frac{dr}{\sqrt{E - V(q)}} \\
&= \sqrt{2m}\int_{r_1(E)}^{r_2(E)} \frac{dr}{\sqrt{E + \sqrt{\alpha k} - kr^2/2 - \alpha/(2r^2)}} \\
&= 2\sqrt{m}\int_{r_1(E)}^{r_2(E)} \frac{r\,dr}{\sqrt{-\alpha + 2r^2(E + \sqrt{k\alpha}) - kr^4}} \\
&= \sqrt{\frac{m}{k}}\int_{x_1}^{x_2} \frac{dx}{\sqrt{(x - x_1)(x_2 - x)}},
\end{aligned}$$

where $x = r^2$, $x_1 = r_1(E)^2$ and $x_2 = r_2(E)^2$. In order to evaluate this last integral one may define a new variable θ, varying between 0 and $\frac{1}{2}\pi$, by

$$x = x_1\cos^2\theta + x_2\sin^2\theta,$$

so that

$$\frac{dx}{d\theta} = 2(x_2 - x_1)\sin\theta\cos\theta$$

and

$$x - x_1 = (x_2 - x_1)\sin^2\theta, \qquad x_2 - x = (x_2 - x_1)\cos^2\theta.$$

In terms of θ the integral becomes

$$T(E) = \sqrt{\frac{m}{k}}\int_0^{\pi/2} 2\,d\theta = \pi\sqrt{\frac{m}{k}}.$$

Notice that this expression is independent of the energy, so that all phase curves have the same period. The reason for this untypical behaviour is simply that the potential $V(r)$ is the effective potential for a two-dimensional linear oscillator; the inverse square term comes from the angular momentum. Another way of describing this motion would be to use Cartesian coordinates (x, y); then the motion in either coordinate direction is unaffected by the motion in the other, and is described by the Hamiltonian $H = \frac{1}{2}p^2 + \frac{1}{2}kz^2$, where z is either x or y.

Solution 4.14

The relative positions of the charges and the moving particle are shown in Figure S4.8.

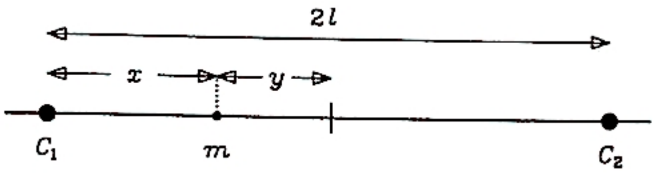


Figure S4.8

From the question we see that the force, in the positive x direction, on the charge is

$$F(x) = \frac{\mu}{x^2} - \frac{\mu}{(2l-x)^2} = -\frac{dV}{dx}.$$

The force is zero at one point only, $x = l$, the mid-point, and either side of this it is directed towards the mid-point. The potential is obtained by integrating $F(x)$,

$$V = \frac{\mu}{x} + \frac{\mu}{2l-x} = \frac{2\mu l}{l^2 - y^2},$$

where y is the distance of the charge from the mid-point of C_1C_2, $y = x - l$, as shown in Figure S4.8; this is a more convenient coordinate, as the potential is then an even function; see Figure S4.9.

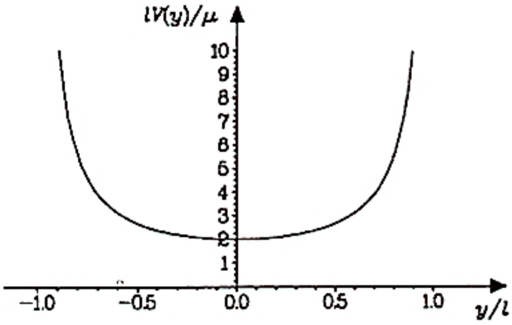


Figure S4.9 Graph of the scaled potential $lV(y)/\mu$ as a function of y/l.

The Hamiltonian is thus

$$H(y,p) = \frac{p^2}{2m} + \frac{2\mu l}{l^2 - y^2}.$$

The potential is even, tends to infinity as $y \to \pm l$, and has only one minimum, at $y = 0$. Thus all motion is bounded and periodic, with the phase curves being nearly elliptical near the origin, but becoming squashed up as the energy increases and the motion gets nearer the charges at either end, where the potential becomes large, and increases rapidly with y. Some representative phase curves are shown in Figure S4.10.

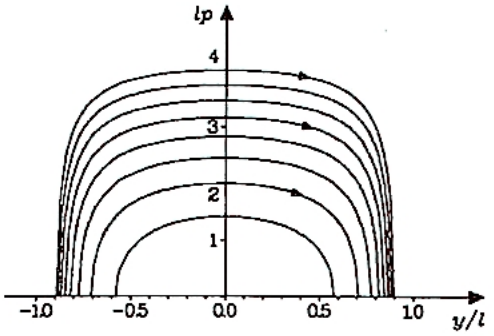


Figure S4.10 Representative phase curves in the upper half-plane for the Hamiltonian $H(y,p) = p^2/2m + 2\mu l/(l^2 - y^2)$.

With the initial conditions given, the energy is

$$E = \frac{2\mu}{l(1-k^2)}.$$

The period is therefore

$$\begin{aligned} T(E) &= 2\sqrt{2m}\int_0^{kl} \frac{dy}{\sqrt{E - 2\mu l/(l^2-y^2)}} \\ &= 2\sqrt{2m}\int_0^{kl} \frac{dy}{\sqrt{2\mu/l(1-k^2) - 2\mu l/(l^2-y^2)}} \\ &= 2\sqrt{\frac{m}{\mu}}\int_0^{kl} dy\,\sqrt{\frac{(1-k^2)(l^2-y^2)}{k^2l^2-y^2}} \\ &= 2\sqrt{\frac{ml^3(1-k^2)}{\mu}}\int_0^{\pi/2} d\theta\,\sqrt{1-k^2\sin^2\theta}, \end{aligned}$$

where the last equation is obtained by putting $y = kl\sin\theta$.

Solution 4.15

If the turning points are $q_1(E) < 0$ and $q_2(E) > 0$, then since the potential is even, $q_1 = -q_2$. (We use q_1 and q_2 instead of $q_1(E)$ and $q_2(E)$ to save writing.) The period may be written in the form

$$\begin{aligned} T(E) &= \sqrt{2m}\int_{-q_2}^{q_2} \frac{dq}{\sqrt{E - V(q)}} \\ &= \sqrt{2m}\left\{\int_0^{q_2} \frac{dq}{\sqrt{E-V(q)}} + \int_{-q_2}^{0} \frac{dq}{\sqrt{E-V(q)}}\right\} \\ &= 2\sqrt{2m}\int_0^{q_2} \frac{dq}{\sqrt{E-V(q)}}, \end{aligned}$$

where the last equation is obtained by putting $q = -x$ in the integrand and using the fact that the potential is even, $V(-x) = V(x)$.

Using the above result we can write the period in the form

$$T(E) = 2\sqrt{2m}\int_0^{q_2} \frac{dq}{\sqrt{E-(q/l)^\alpha}}, \qquad q_2 = lE^{1/\alpha}.$$

Now put $q/l = E^{1/\alpha}y$ in order to remove the dependence of both l and E from the integral:

$$T(E) = \left(2\sqrt{2m}\right) lE^{1/\alpha - 1/2}\int_0^1 \frac{dy}{\sqrt{1-y^\alpha}}.$$

Next, put $x = y^\alpha$, or $y = x^{1/\alpha}$, to rewrite this integral in the form

$$T(E) = \left(2\sqrt{2m}\right) l E^{(2-\alpha)/2\alpha} \int_0^1 dx\, \frac{(1/\alpha)x^{(1/\alpha-1)}}{\sqrt{1-x}}$$
$$= \frac{2l\sqrt{2m}}{\alpha E^{(\alpha-2)/2\alpha}}\, J(\alpha),$$

where $J(\alpha)$ is defined in the question.

You may be wondering why the last transformation was made, as it does not seem to have made the integral any easier. In fact, this last form of the integral is one of the common integral forms of a standard function known as the Beta function, for which tabulated values and computer routines exist, so it is more convenient to have the answer in this form.

Solution 4.16

(i) The graph of the potential function is shown in Figure S4.11.

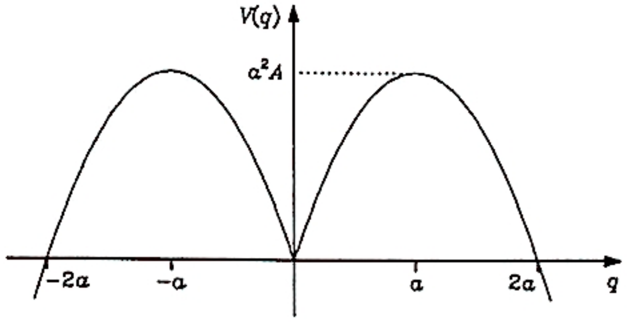


Figure S4.11 Graph of the even potential with $V(q) = Aq(2a - q)$ for $q > 0$.

In the range $q > 0$ the potential has one stationary point, at the root of

$$V'(q) = 2A(a - q) = 0, \qquad \text{that is,} \quad q = a,$$

and here $V''(a) < 0$ so $q = a$ is a maximum of the potential with value $V(a) = Aa^2$. There is also a minimum at $q = 0$, but near here $V(q) = 2aA|q|$ (see Figure S4.11), and this is not an ordinary minimum. From the graph of this potential we see that there is librational motion for energies in the range $0 < E < Aa^2$, that the separatrix energy is $E_s = Aa^2$, that unbounded motion occurs for *all* energies, but for $E > E_s$ and $E < 0$ there is only unbounded motion. Some representative phase curves are shown in Figure S4.12, where the dotted line is the separatrix. Notice that the phase curves have discontinuous gradient at $q = 0$, because the potential gradient, that is the force, is discontinuous here. The separatrix consists of segments of the straight lines given by

$$p^2 = 2m\left(Aa^2 - Aq(2a - q)\right) = 2mA(a - q)^2$$

or $\quad p(q) = \pm\sqrt{2mA}\,(q - a) \qquad \text{for } q > 0.$

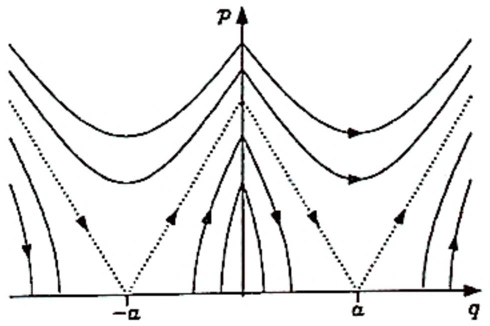


Figure S4.12 Representative phase curves for the even potential of Figure S4.11. The system is even in p, so only the upper half-plane is shown.

(ii) Using the result of the previous question, the period is

$$T(E) = 2\sqrt{2m}\int_0^{q_2} \frac{dq}{\sqrt{E - Aq(2a-q)}}.$$

We now set $x = q/a$ and $\varepsilon = E/Aa^2$. This has the advantage of making the integrand dimensionless, and the separatrix energy is given by $\varepsilon = 1$. The expression for the period becomes

$$T(\varepsilon) = 2\sqrt{\frac{2m}{A}}\int_0^{x_2} \frac{dx}{\sqrt{(1-x)^2 - (1-\varepsilon)}}.$$

Now put $1 - x = \sqrt{1-\varepsilon}\cosh\theta$ (note that θ decreases with increasing x) to give

$$T(\varepsilon) = -2\sqrt{\frac{2m}{A}}\int_{\theta_1}^{\theta_2} d\theta,$$

where $\cosh\theta_2 = 1$, or $\theta_2 = 0$, and $\cosh\theta_1 = (1-\varepsilon)^{-1/2}$. Then

$$\begin{aligned}
T(\varepsilon) &= 2\sqrt{\frac{2m}{A}}\cosh^{-1}(1-\varepsilon)^{-1/2}\\
&= 2\sqrt{\frac{2m}{A}}\ln\left(\frac{1+\sqrt{\varepsilon}}{\sqrt{1-\varepsilon}}\right)\\
&= \sqrt{\frac{2m}{A}}\ln\left(\frac{(1+\sqrt{\varepsilon})^2}{1-\varepsilon}\right)\\
&= \sqrt{\frac{2m}{A}}\ln\left(\frac{1+\sqrt{\varepsilon}}{1-\sqrt{\varepsilon}}\right),
\end{aligned}$$

which is the required relation.

The behaviour of the period of small-amplitude oscillations for this system is interesting. Small-amplitude oscillations correspond to small energies, that is, to small ε. The expansion

$$\ln\left(\frac{1+x}{1-x}\right) = 2x + O(x^3)$$

can be used to give the approximation

$$T(\varepsilon) = 2\sqrt{\frac{2m}{A}}\sqrt{\varepsilon} \qquad \text{or} \qquad T(E) = \frac{2\sqrt{2mE}}{Aa},$$

for the period. This is linear in $\sqrt{E}$, whereas most small-amplitude oscillations have period which is independent of the energy. The reason for this unusual result is simply that for $q \ll a$ the potential is given approximately by $V(q) \simeq 2aA|q|$, so the motion does not behave like a 'normal' system for which the potential would be proportional to q^2 at the minimum. Notice that this is a special case of the previous exercise with $\alpha = 1$ and $l = 1/Aa$.

(iii) In the dimensionless units, the separatrix has energy $\varepsilon = 1$, so to find the behaviour of the period near the separatrix we put $\varepsilon = 1 - \delta$, $\delta > 0$. Then $1 + \sqrt{\varepsilon} \simeq 2$ and

$$1 - \sqrt{\varepsilon} = 1 - \sqrt{1-\delta} \simeq 1 - (1 - \tfrac{1}{2}\delta) = \tfrac{1}{2}\delta,$$

so

$$T(\varepsilon) = \sqrt{\frac{2m}{A}}\ln\left(\frac{4}{\delta}\right), \qquad \delta = 2(1-\sqrt{\varepsilon}), \tag{S4.1}$$

which diverges logarithmically as $\delta \to 0$, or $\varepsilon \to 1$.

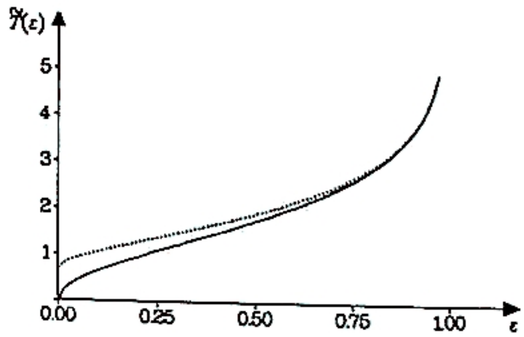


Figure S4.13 Graph of the scaled period $\tilde{T}(\epsilon) = \sqrt{A/2m}T(\epsilon)$ as a function of the scaled energy ϵ. The dotted line is the approximation S4.1.

A graph of the period as a function of the energy is shown in Figure S4.13, where we also show the approximation of Equation S4.1, which is seen to be quite good for $\epsilon \simeq 1$.

The result of Equation S4.1 has important applications. The period of any periodic motion near the separatrix produced by a parabolic barrier behaves as $-\ln(E_s - E)$, as E tends towards E_s, the separatrix energy, because most of the time on the orbit is spent near the saddle, as here the phase point is moving slowest. Thus an approximation to the motion near this point provides a reasonable approximation to the whole period provided that $|E - E_s|$ is sufficiently small. The importance of this general result is that these long-period orbits are the most susceptible to small perturbations, simply because on them a small perturbation has time to accumulate to produce a significant effect. We shall see examples of this later in the course.

Solution 4.17

For motion in a circle, the configuration coordinate q varies in the range $0 \le q \le 2\pi$, with the ends of the range, $q = 0,\ 2\pi$, labelling the same physical point. The conjugate momentum p can usually take any value in $(-\infty, \infty)$, positive values of p corresponding to increasing q, and vice versa. The phase space can then be represented as an infinitely long cylinder with p increasing up the cylinder axis.

Motion in a circle is discussed in Section 4.6. The treatment of the phase space as a cylinder is explained in Subsection 4.6.1.

However, this is not very convenient for graphical representation on two-dimensional paper, so the cylinder is usually unwrapped onto the plane. If the cylinder is cut along the line $q = \pi$, then the two vertical lines $q = \pm\pi$ correspond to the same physical configuration: a phase curve leaving the strip $-\pi \le q \le \pi$ on one side returns on the other side with the same p-coordinate as it had when it left. The cylinder need not be cut at π; any line of constant q will do. Sometimes it is more convenient to make the cut at $q = 2\pi$ and show the range $0 \le q < 2\pi$. In this case the identification of corresponding points of opposite vertical sides of the strip is equivalent to the identification of $q = 0$ and $q = 2\pi$ mentioned above.

If the motion of a system is periodic and the generalised coordinate q is expressed as a function of time, then the motion is a rotation if $\dot{q}$ always has the same sign, that is, if $\dot{q}$ is never zero; it is a libration if $\dot{q}$ changes sign.

Rotations and librations are distinguished in Subsection 4.6.3.

Every motion of a free rotor other than a fixed point is a rotation. The small oscillations of a vertical pendulum are librations, while those of its motions for which there is sufficient energy for the pendulum to pass through the vertical are rotations. Librations can occur when the phase space is a plane: any bound, periodic motion in a potential well is a libration. Rotations, on the other hand, are possible only if the configuration space is periodic in the generalised coordinate.

Solution 4.18

The fixed points of this system are at the roots of the equations

$$\frac{\partial H}{\partial q} = p\cos q = 0, \qquad \frac{\partial H}{\partial p} = p + \sin q = 0.$$

From the first, either $p = 0$ or $q = \pm\frac{1}{2}\pi$. When $p = 0$, the second equation gives $q = 0$ or π, and when $q = \pm\frac{1}{2}\pi$ it gives $p = \mp 1$. The nature of each of these fixed points is established by finding the second derivatives of the Hamiltonian; the results are summarised in the table below.

Fixed point	$\dfrac{\partial^2 H}{\partial q^2}$	$\dfrac{\partial^2 H}{\partial p^2}$	$\dfrac{\partial^2 H}{\partial p\partial q}$	Type of stationary point	Type of fixed point
	$-p\sin q$	1	$\cos q$		
$(0,0)$	0	1	1	saddle	saddle
$(\pi,0)$	0	1	-1	saddle	saddle
$(\frac{1}{2}\pi,-1)$	1	1	0	minimum	centre
$(-\frac{1}{2}\pi,1)$	1	1	0	minimum	centre

The equations of motion are

$$\dot{p} = -p\cos q, \qquad \dot{q} = p + \sin q.$$

The line $p = 0$ is a phase curve; it joins the unstable fixed points $(0,0)$ and $(\pi,0)$, as shown in Figure S4.14. It is one branch of the separatrix; the energy of the separatrix is $E_s = 0$, so its other branch is

$$p_s(q) = -2\sin q.$$

Thus the phase space is divided into two regions of libration, around each stable fixed point, and two regions of rotational motion, one for positive p, the other for negative p. These are shown in Figure S4.14.

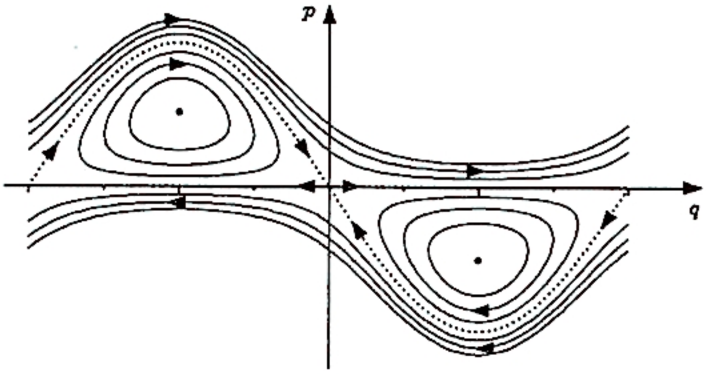


Figure S4.14 Representative phase curves of the Hamiltonian $H(q,p) = \frac{1}{2}p^2 + p\sin q$.

Consider the fixed point at $(-\frac{1}{2}\pi, 1)$, which has energy $E = -\frac{1}{2}$. A librating phase curve in the same region has two branches, which are the roots of

$$\tfrac{1}{2}p^2 + p\sin q = E, \qquad -\tfrac{1}{2} < E < 0,$$

that is,

$$p_\pm(q) = -\sin q \pm \sqrt{\sin^2 q - 2|E|}.$$

It is now convenient to move the q-origin by defining $x = q + \frac{1}{2}\pi$, so that the fixed point is at $x = 0$. Then $\sin q = -\cos x$ and

$$p_\pm(x) = -\cos x \pm \sqrt{\cos^2 x - 2|E|},$$

with $-\alpha \le x \le \alpha$, where $\cos^2\alpha = 2|E|$ and $0 < \alpha < \frac{1}{2}\pi$. The total time taken to pass once round a phase curve is the sum of the times along each branch,

$$T(E) = \int_0^T dt = \int_{-\alpha}^{\alpha} \frac{dx}{\dot{x}_+(x)} + \int_{\alpha}^{-\alpha} \frac{dx}{\dot{x}_-(x)},$$

where $\dot{x}_+$ denotes the value of $\dot{x}$ on the upper branch and $\dot{x}_-$ its value on the lower one. But from Hamilton's equations,

$$\dot{x}_\pm = \dot{q}_\pm = p_\pm - \cos x = \pm\sqrt{\cos^2 x - 2|E|},$$

so that

$$T(E) = 2\int_{-\alpha}^{\alpha} \frac{dx}{\sqrt{\cos^2 x - 2|E|}} = 4\int_0^{\alpha} \frac{dx}{\sqrt{\cos^2 x - 2|E|}},$$

the last step following from the fact that the integrand is an even function of x.

Solution 4.19

The fixed points are at the roots of

$$\frac{\partial H}{\partial p} = p = 0, \quad \frac{\partial H}{\partial q} = Fn\sin nq = 0, \qquad \text{that is,} \quad nq = m\pi.$$

This system is rather like the vertical pendulum, treated in Subsection 4.6.2.

So there are n fixed points, at $(m\pi/n, 0)$, $m = 0, 1, \ldots, (2n-1)$. In order to classify these we form the second derivatives

$$\frac{\partial^2 H}{\partial p^2} = 1, \qquad \frac{\partial^2 H}{\partial p \partial q} = 0, \qquad \frac{\partial^2 H}{\partial q^2} = Fn^2 \cos nq = Fn^2 \cos m\pi.$$

For a fixed point to be stable it is necessary that $\partial^2 H/\partial q^2 > 0$, so for a stable fixed point we need $\cos m\pi > 0$, and the stable points correspond to $m = 0, 2, \ldots$. Similarly, the unstable fixed points are at $m = 1, 3, \ldots$.

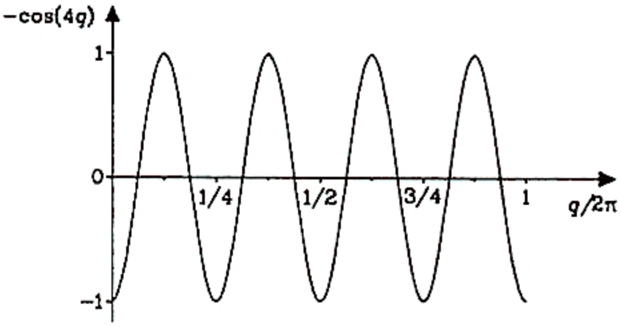


Figure S4.15 Graph of the potential $V(q) = -\cos 4q$.

This analysis is confirmed in Figure S4.15, where we have plotted the potential $V(q) = -F\cos 4q$, having maxima at $q = \frac{1}{4}(2k+1)\pi$, $k = 0, 1, 2, 3$, giving unstable fixed points, and minima at $q = \frac{1}{2}k\pi$, $k = 0, 1, 2, 3$, giving stable fixed points.

In each of the wells the motion will be librational, so that for $E < F$ all motion is librational and for $E > F$ all motion is rotational, with the separatrix at $E = E_s = F$ joining the unstable fixed points and separating these two types of motion. No motion is possible for $E < -F$. Some representative phase curves are shown in Figure S4.16.

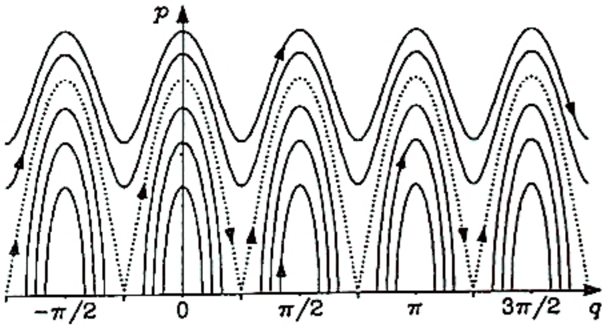


Figure S4.16 Phase curves in the $p > 0$ half-plane for the Hamiltonian $H(q, p) = \frac{1}{2}p^2 - F\cos 4q$.

Solution 4.20

(i) The general expression for the period of rotational motion is

$$T(E) = \sqrt{\frac{m}{2}} \int_0^{2\pi} dq \, \frac{1}{\sqrt{E - V(q)}}.$$

But we also have

$$\frac{\partial}{\partial E} \left(\sqrt{E - V(q)} \right) = \frac{1}{2\sqrt{E - V(q)}},$$

and on substituting this identity into the integral for the period we obtain

$$T(E) = \sqrt{2m} \int_0^{2\pi} dq \, \frac{\partial}{\partial E} \left(\sqrt{E - V(q)} \right).$$

Differentiation with respect to E and integration with respect to q commute, so

$$T(E) = \sqrt{2m} \frac{d}{dE} \int_0^{2\pi} dq \, \sqrt{E - V(q)}.$$

The derivative at the final stage is an ordinary, not a partial, derivative, because after integration we are left with a function of E alone.

(ii) For the modified potential $V_0 + \varepsilon V_1$ the result of part (i) gives the following formula for the period:

$$T(E) = \sqrt{2m} \frac{d}{dE} \int_0^{2\pi} dq \, \sqrt{E - V_0(q) - \varepsilon V_1(q)}.$$

Thus the difference between the periods for the two potentials is

$$T_1(E) = \sqrt{2m} \frac{d}{dE} \int_0^{2\pi} dq \, \left(\sqrt{E - V_0(q) - \varepsilon V_1(q)} - \sqrt{E - V_0(q)} \right).$$

Now we expand the integrand in powers of ε to obtain

$$\begin{aligned} \sqrt{E - V_0 - \varepsilon V_1} &= \sqrt{E - V_0} \left(1 - \frac{\varepsilon V_1}{E - V_0} \right)^{1/2} \\ &\simeq \sqrt{E - V_0} - \frac{\varepsilon V_1}{2\sqrt{E - V_0}} + O(\varepsilon^2). \end{aligned}$$

Thus the difference in the periods may be written in the form

$$T_1(E) = -\varepsilon \sqrt{\frac{m}{2}} \frac{d}{dE} \int_0^{2\pi} dq \, \frac{V_1(q)}{\sqrt{E - V_0(q)}}$$

to first order in ε. But on a phase curve of the V_0 motion we have

$$\dot{q} = \frac{p}{m} = \sqrt{\frac{2}{m}} \sqrt{E - V_0(q)},$$

and so

$$T_1(E) = -\varepsilon \frac{d}{dE} \int_0^{2\pi} dq \, \frac{V_1(q)}{\dot{q}} = -\varepsilon \frac{d}{dE} \int_0^{T_0} dt \, V_1(q(t))$$

on changing the integration variable from q to t. This expresses the change in the period as an integral along a trajectory of the motion due to V_0.

This is an important result. It can be generalised to librational motion, provided care is taken with the limits of integration; we shall see similar results later in the course.

Solution 4.21

At the angle $q = -\alpha$ the momentum suddenly changes sign but not magnitude. So the easiest way to draw the phase curves is by plotting the phase curves for the ordinary vertical pendulum and inserting barriers at $q = -\alpha$ and $q = \pi$. This is done in Figure S4.17; the barriers are shown by solid vertical lines, and the excluded region of the phase space by shading.

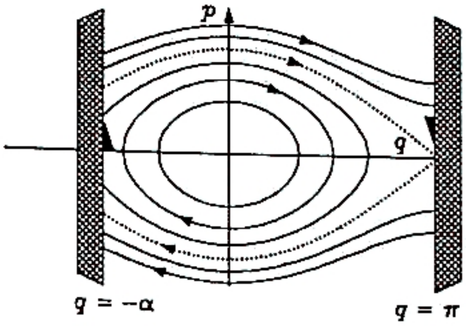


Figure S4.17

For all phase curves not intersecting a barrier, the motion is unaffected by the barriers so is the same as if they were not present. But for larger-amplitude motion, at the intersection of a lower phase curve with the barrier $q = -\alpha$ the phase point moves vertically up to the corresponding upper phase curve, so that there is an instantaneous transition from the point $(-\alpha, -p)$ to the point $(-\alpha, p)$, as shown in the figure.

For small-amplitude motion, when the barrier has no effect, the period is given by the integral expression

$$T(E) = l\sqrt{2m}\int_{q_1}^{q_2} \frac{dq}{\sqrt{E + mgl\cos q}},$$

where m is the mass of the pendulum bob, and we have assumed that the external force is that due to gravity. Here $q_1 < 0$ and $q_2 > 0$ are the two turning points, and we are assuming that $q_1 > -\alpha$. In this case $q_1 = -q_2$, so the period may also be written in the form

$$T(E) = 2l\sqrt{2m}\int_{0}^{q_2} \frac{dq}{\sqrt{E - a\cos q}}, \qquad \text{with} \quad \cos q_2 = E/a.$$

If $E > a\cos\alpha$, the barrier at $q = -\alpha$ is hit, and the parts of the motion with $q > 0$ and $q < 0$ must be treated separately. We then have

$$T(E) = l\sqrt{\frac{m}{2}}\left(\int_{0}^{q_2} \frac{dq}{\sqrt{E + mgl\cos q}} + \int_{0}^{\alpha} \frac{dq}{\sqrt{E + mgl\cos q}}\right),$$

with $\alpha \le q_2 \le \pi$, where we have used the fact that $\cos q$ is an even function to change the last integral to the range $(0, \alpha)$, rather than $(-\alpha, 0)$.

If the energy is sufficiently large that both barriers are hit, that is $E \ge mgl$, then the upper turning point is at $q = \pi$ so the period is given by the above integral with q_2 replaced by π.

Solution 4.22

This is a rather unusual Hamiltonian, but it is not completely artificial as it is a simple version of the Hamiltonian describing an atom with a singly excited electron in a weak, static electric field.

It is rather a difficult system to analyse as it has some peculiar features caused by the square root singularity at $p = \pm 1$, the origin of which will become clear later on. Since the Hamiltonian is not of mechanical type, we cannot find the fixed points by simply analysing a potential function. To obtain some idea of the shape of the surface (in three dimensions) defined by the Hamiltonian, it is easiest to first consider the function

$$f(p) = p + s\sqrt{1 - p^2}$$

for some constant s. Different values of s give the sections of the Hamiltonian surface corresponding to different values of q, with $s = \sin q$.

The derivative $f'(p)$ is singular at $p = \pm 1$, and

$$f'(p) = 1 - \frac{sp}{\sqrt{1-p^2}} = 0 \quad \text{at} \quad p = \begin{cases} (1+s^2)^{-1/2}, & s > 0, \\ -(1+s^2)^{-1/2}, & s < 0. \end{cases}$$

Graphs of $f(p)$ are given in Figure S4.18 for $s > 0$, $s = 0$ and $s < 0$; notice that both branches of the curve produce an ellipse.

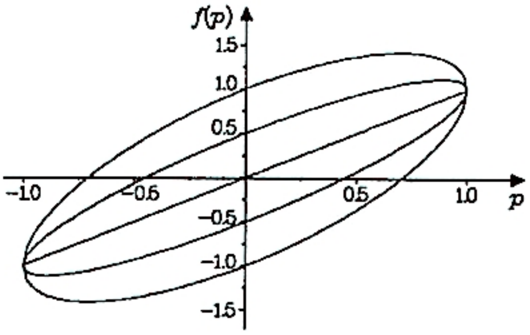


Figure S4.18 Graphs of the function $f(p) = p + s\sqrt{1-p^2}$, for $s = 0, \pm\frac{1}{2}, \pm 1$.

For fixed $p \neq \pm 1$, on the other hand, we obtain simply a sine curve; in the range $(-\pi, \pi)$, $\sin q$ is stationary at $q = \pm\frac{1}{2}\pi$. If you have sufficient geometrical imagination you may find it possible to visualise the Hamiltonian surface from this information.

It follows that $H(q,p)$ has two stationary points for p, q in their given ranges: a maximum where $q = \frac{1}{2}\pi$, $s = 1$ and $p = \frac{1}{\sqrt{2}}$, and a minimum where $q = -\frac{1}{2}\pi$, $s = -1$ and $p = -\frac{1}{\sqrt{2}}$. That is, there are two fixed points, at $(\frac{1}{2}\pi, \frac{1}{\sqrt{2}})$ and $(-\frac{1}{2}\pi, -\frac{1}{\sqrt{2}})$; both of these points are centres. The corresponding energies are $\pm\sqrt{2}$.

On a phase curve with energy E we have

$$\sin q = \frac{E-p}{\sqrt{1-p^2}},$$

whence

$$\dot{q} = \frac{\partial H}{\partial p} = 1 - \frac{p \sin q}{\sqrt{1-p^2}} = \frac{1-Ep}{1-p^2}.$$

It follows that on phase curves with $-1 < E < 1$ there is no value of p such that $-1 < p < 1$ for which $\dot{q} = 0$; motion on such phase curves is therefore a rotation. The phase curves with energy E such that $1 < E < \sqrt{2}$ or $-\sqrt{2} < E < -1$, on the other hand, give librational motion in the region of one or other of the fixed points. There are therefore two separatrixes, with energies $E_s = \pm 1$, which divide the regions where motion is librational from those where it is rotational. However, these are not normal separatrixes: they are not associated with any saddle points.

Consider the separatrix with $E = 1$. It has two branches, the line $p = 1$, and the curve

$$p + \sqrt{1-p^2}\sin q = 1 \qquad \text{or} \qquad p_s(q) = \frac{1-\sin^2 q}{1+\sin^2 q} \qquad \text{for} \quad q \geq 0;$$

the restriction on q is required in order that the left-hand equation be satisfied with $p < 1$. Thus $p_s(\frac{1}{2}\pi) = 0$ and $p_s(0) = p_s(\pi) = 1$; the two branches of the separatrix enclose the fixed point at $(\frac{1}{2}\pi, \frac{1}{\sqrt{2}})$. The separatrix with $E = -1$ likewise has two branches, one of which is the line $p = -1$; they enclose the fixed point at $(-\frac{1}{2}\pi, -\frac{1}{\sqrt{2}})$. Apart from these two separatrixes, the general shape of the phase curves is clear; the phase diagram is shown in Figure S4.19.

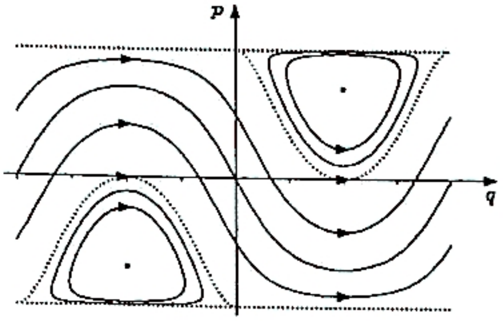


Figure S4.19 Representative phase curves for the Hamiltonian $H(q,p) = p + \sqrt{1-p^2}\sin q$.

The facts that the branches of the separatrixes intersect, and that they are not associated with saddles, are distinctly odd: what does this mean? Consider the equations of motion

$$\dot{p} = -\frac{\partial H}{\partial q} = -\sqrt{1-p^2}\cos q, \qquad \dot{q} = \frac{\partial H}{\partial p} = 1 - \frac{p\sin q}{\sqrt{1-p^2}}.$$

On the line $p = 1$ we have $\dot{p} = 0$, but $\dot{q}$ is unbounded; in other words, the phase point moves from $q = \pi$ to $q = 0$ in no time. We cannot therefore regard the points on the line $p = 1$ as physically distinct. A similar comment applies to the line $p = -1$.

This situation seems rather strange, but in fact there is a very natural explanation for it, which can best be described by reference to another problem, that of representing a map of the sphere on two-dimensional paper. The problem there arises in dealing with the poles, and it is a similar difficulty that we face here. In fact, the system under discussion should really be regarded as one whose phase space is a sphere; the problem we have noticed arises simply because of the inadequacy of the plane as a representation of the surface of a sphere. Suppose we first wrap the phase plane round to form a phase cylinder, in the usual way for rotational motion, but then squeeze the lines $p = \pm 1$ into points to form a spherical phase space. Thus all the points $(q, 1)$ and $(q, -1)$ of the cylinder become respectively the north and south poles of the sphere. The problems we encountered above disappear; there is no longer any distinction between librational and rotational motion, and the curves we have called separatrixes become ordinary phase curves which happen to pass through the poles. We can see that the system is well-behaved at the poles by transforming to suitable variables and expanding the equation of motions about a pole. One possible variable is $p = \cos\psi$, as illustrated in Figure S4.20.

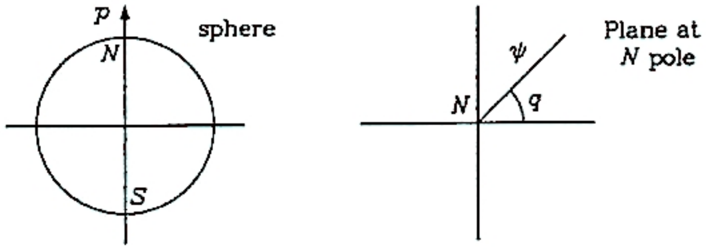


Figure S4.20

For motion near the north pole N we assume that ψ is small; noting that near N, ψ is just the radial distance from the pole and q the usual polar angle, we can convert to local Cartesian coordinates $x = \psi\cos q$, $y = \psi\sin q$, to give the well-behaved equations of motion

$$\dot{x} = 1 - y, \qquad \dot{y} = x.$$

This is another example of how the topology of phase space can affect the types of motion possible, although spherical phase spaces seem rather rare.

ISBN 0 7492 4777 0

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